[10] R. Onvural, "Asynchronous Transfer Mode Networks: Performance Issues", Artech House, Norwood, MA (1994).

[11] J. Patel, "Performance of Processor Memory Interconnections for Multiprocessors", *IEEE Transactions on Computers*, **C-30**(10), pp. 771-780 (October 1981).

[12] H. Takagi, *Queueing Analysis, A Foundation of Performance Evaluation, Volume 1: Vacation and Priority Systems, Part 1*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands (1991).

[13] H. Takagi, *Queueing Analysis, A Foundation of Performance Evaluation, Volume 3: Discrete-Time Systems*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands (1993).

$$= \sum_{i=0}^{M} \left(\begin{array}{c} M \\ i \end{array} \right) \lambda^{M-i}(1-\lambda)^i - \sum_{i=0}^{M} \left(\begin{array}{c} M \\ i \end{array} \right) [\lambda(1-\frac{1}{N})]^{M-i}(1-\lambda)^i = 1-(1-\frac{\lambda}{N})^N \qquad \Box.$$

References

[1] T. Chen and S. Liu, *ATM Switching Systems*, Artech House, Norwood, MA (1995)

[2] M. G. Hluchyj and M. J. Karol, "Queueing in High-Performance Packet Switching", *IEEE Journal on Selected Areas in Communications*, **6**(9), pp. 1587-97 (December 1988).

[3] J. Y. Hui and E. Arthurs, "A Broadband Packet Switch for Integrated Transport", *IEEE Journal on Selected Areas in Communications*, **SAC-5**(8), pp. 1264-73 (October 1987).

[4] J. Y. Hui, *Switching and Traffic Theory for Integrated Broadband Networks*. Norwell, MA: Kluwer Academic Publishers (1991).

[5] M. J. Karol, M. G. Hluchyj and S. P. Morgan, "Input versus Output Queueing on a Space-Division Packet Switch", *IEEE Transactions on Communications*, **COM-35**(12), pp. 1347-56 (December 1987).

[6] C. Kolias and L. Kleinrock, "The Odd-Even Input-Queueing ATM Switch: Performance Evaluation", *International Conference on Communications (ICC) '96*, pp. 1674-1679 (June 1996).

[7] C. Kolias and L. Kleinrock, "Yet another analytic study of input-buffered ATNM switches", *Technical Report*, Computer Science Department, UCLA (April 1997).

[8] L. Kleinrock, *Queueing Systems, Volume 1: Theory*, Wiley-Interscience, New York (1975).

[9] L. Kleinrock, *Queueing Systems, Volume 2: Computer Applications*, Wiley-Interscience, New York (1976).

$$\sum_{k=0}^{N} d^*(k)\frac{k}{N} = 1-(1-\frac{1}{N})^M$$

We have

$$\begin{aligned}
\sum_{k=0}^{N} d^*(k)\frac{k}{N} &= \sum_{k=0}^{N}\left(\begin{array}{c} N \\ k \end{array}\right)\frac{m_k}{N^M}\frac{k}{N} = \frac{m_N}{N^M} + \sum_{k=0}^{N-1}\left(\begin{array}{c} N \\ k \end{array}\right)\frac{km_k}{N^{M+1}} \\
&= \frac{N^M}{N^M} - \sum_{j=1}^{N-1}\left(\begin{array}{c} N \\ j \end{array}\right)\frac{m_j}{N^M} + \sum_{k=1}^{N-1}\left(\begin{array}{c} N \\ k \end{array}\right)\frac{km_k}{N^{M+1}} = 1 - \sum_{k=1}^{N-1}(N-k)\left(\begin{array}{c} N \\ k \end{array}\right)\frac{m_k}{N^{M+1}} \\
&= 1 - \sum_{k=1}^{N-1}\left(\begin{array}{c} N-1 \\ k \end{array}\right)\frac{m_k}{N^M} = 1 - \frac{m_{N-1}}{N^M} - \sum_{k=1}^{N-2}\left(\begin{array}{c} N-1 \\ k \end{array}\right)\frac{m_k}{N^M} \\
&= 1 - \frac{(N-1)^M}{N^M} + \sum_{k=1}^{N-2}\left(\begin{array}{c} N-1 \\ k \end{array}\right)\frac{m_k}{N^M} - \sum_{k=1}^{N-2}\left(\begin{array}{c} N-1 \\ k \end{array}\right)\frac{m_k}{N^M} = 1-(1-\frac{1}{N})^M \qquad \square.
\end{aligned}$$

Now, in proving Equation (49)

$$\begin{aligned}
\gamma &= \sum_{k=0}^{N} d(k)\frac{k}{N} \\
&= \sum_{k=0}^{N} d^*(k)\lambda^M\frac{k}{N} + \sum_{k=0}^{N}\sum_{i=1}^{M}\left(\begin{array}{c} M \\ i \end{array}\right)\left(\begin{array}{c} N \\ k \end{array}\right)\lambda^{M-i}(1-\lambda)^i\frac{k\beta_{i,k}}{N^{M+1-i}} \\
&= [1-(1-\frac{1}{N})^M]\lambda^M + \sum_{i=1}^{M}\left(\begin{array}{c} M \\ i \end{array}\right)\lambda^{M-i}(1-\lambda)^i\sum_{k=1}^{N}\left(\begin{array}{c} N \\ k \end{array}\right)\frac{k\beta_{i,k}}{N^{M+1-i}} \\
&= [1-(1-\frac{1}{N})^M]\lambda^M + \sum_{i=1}^{M}\left(\begin{array}{c} M \\ i \end{array}\right)\lambda^{M-i}(1-\lambda)^i[N\frac{\beta_{i,N}}{N^{M+1-i}} + \sum_{k=1}^{N-1}\left(\begin{array}{c} N \\ k \end{array}\right)\frac{k\beta_{i,k}}{N^{M+1-i}}] \\
&= [1-(1-\frac{1}{N})^M]\lambda^M + \sum_{i=1}^{M}\left(\begin{array}{c} M \\ i \end{array}\right)\lambda^{M-i}(1-\lambda)^i[N\frac{N^{M-i}}{N^{M+1-i}} - \sum_{j=1}^{N-1}N\left(\begin{array}{c} N \\ j \end{array}\right)\frac{\beta_{i,j}}{N^{M+1-i}}] \\
&\quad + \sum_{k=1}^{N-1}k\left(\begin{array}{c} N \\ k \end{array}\right)\frac{\beta_{i,k}}{N^{M+1-i}}] \\
&= [1-(1-\frac{1}{N})^M]\lambda^M + \sum_{i=1}^{M}\left(\begin{array}{c} M \\ i \end{array}\right)\lambda^{M-i}(1-\lambda)^i[1 - \sum_{j=1}^{N-1}(N-k)\left(\begin{array}{c} N \\ k \end{array}\right)\frac{\beta_{i,j}}{N^{M+1-i}}] \\
&= [1-(1-\frac{1}{N})^M]\lambda^M + \sum_{i=1}^{M}\left(\begin{array}{c} M \\ i \end{array}\right)\lambda^{M-i}(1-\lambda)^i[1 - \sum_{k=1}^{N-1}\left(\begin{array}{c} N-1 \\ k \end{array}\right)\frac{\beta_{i,k}}{N^{M-i}}] \\
&= [1-(1-\frac{1}{N})^M]\lambda^M + \sum_{i=1}^{M}\left(\begin{array}{c} M \\ i \end{array}\right)\lambda^{M-i}(1-\lambda)^i[1 - \frac{\beta_{i,N-1}}{N^{M-i}} - \sum_{k=1}^{N-2}\left(\begin{array}{c} N-1 \\ k \end{array}\right)\frac{\beta_{i,k}}{N^{M-i}}] \\
&= [1-(1-\frac{1}{N})^M]\lambda^M + \sum_{i=1}^{M}\left(\begin{array}{c} M \\ i \end{array}\right)\lambda^{M-i}(1-\lambda)^i[1 - \frac{(N-1)^{M-i}}{N^{M-i}} + \sum_{k=1}^{N-2}\left(\begin{array}{c} N-1 \\ k \end{array}\right)\frac{\beta_{i,k}}{N^{M-i}} \\
&\quad - \sum_{k=1}^{N-2}\left(\begin{array}{c} N-1 \\ k \end{array}\right)\frac{\beta_{i,k}}{N^{M-i}}] = [1-(1-\frac{1}{N})^M]\lambda^M + \sum_{i=1}^{M}\left(\begin{array}{c} M \\ i \end{array}\right)\lambda^{M-i}(1-\lambda)^i[1-(1-\frac{1}{N})^{M-i}]
\end{aligned}$$

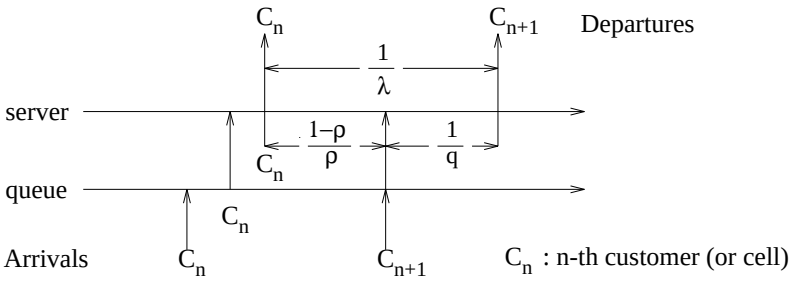

Figure 3: $Geom(\lambda)/Geom(q)/1$ arrival-departure diagram.

where due to the memoryless property, $p_0\frac{1}{\lambda}$ represents the expected number of time slots until the next cell arrival to the system (a cell to find the queue empty will go into service immediately). For a $Geom(\lambda)/G/1$ Equation (B.17) can be alternatively expressed as

$$\frac{1}{\lambda} = \frac{1-\rho}{\rho} + \overline{S} \tag{B.19}$$

where $\overline{S}$ is the mean service time.

Appendix C

Here we present the proofs of Equations (45) and (49).

Proof of Equation (45)

$$\begin{aligned}
\sum_{k=0}^{N} d(k) &= \sum_{k=1}^{N}[\sum_{i=0}^{M}\binom{N}{k}\binom{M}{i}\lambda^{M-i}(1-\lambda)^i\frac{\beta_{i,k}}{N^{M-i}}] \\
&= \lambda^M\sum_{k=1}^{N} d^*(k) + \sum_{i=1}^{M}\binom{M}{i}\lambda^{M-i}(1-\lambda)^i\sum_{k=1}^{N}\binom{N}{k}\frac{\beta_{i,k}}{N^{M-i}} \\
&= \lambda^M + \sum_{i=1}^{M}\binom{M}{i}\lambda^{M-i}(1-\lambda)^i[\binom{N}{N}\frac{\beta_{i,N}}{N^{M-i}} + \sum_{k=1}^{N-1}\binom{N}{k}\frac{\beta_{i,k}}{N^{M-i}}] \\
&= \lambda^M + \sum_{i=1}^{M}\binom{M}{i}\lambda^{M-i}(1-\lambda)^i[\frac{N^{M-i}}{N^{M-i}} - \sum_{j=1}^{N-1}\binom{N}{j}\frac{\beta_{i,j}}{N^{M-i}} + \sum_{k=1}^{N-1}\binom{N}{k}\frac{\beta_{i,k}}{N^{M-i}}] \\
&= \sum_{i=0}^{M}\binom{M}{i}\lambda^{M-i}(1-\lambda)^i = 1 \qquad \square.
\end{aligned}$$

Proof of Equation (49)

First, we show that

distributions, namely $A(z)B(z)$. Taking into consideration that $\mathbf{Pr}[\text{queue is empty}] = 1-r = 1-\frac{\lambda}{q}$ then Equation (B.12) leads to

$$\begin{aligned} E(z) &= r\,A(z)\,B(z) + (1-r)\,B(z) \\ &= \frac{\lambda}{q}\,\frac{\lambda\,z}{1-(1-\lambda)z}\,\frac{qz}{1-1(1-q)z} + \frac{1-\lambda}{q}\,\frac{q\,z}{1-(1-q)z} \end{aligned}$$

After a few calculations we have

$$E(z) = \frac{\lambda\,z}{1-(1-\lambda)z}$$

from which we get

$$e(k) = (1-\lambda)^{k-1}\,\lambda, \qquad\qquad k > 0$$

meaning that the interdeparture time distribution is geometric, with parameter λ, and it is exactly the same as the interarrival distribution. This last result should not come as a surprise: the same conclusion has been drawn for an $M/M/1$ system [8]; as a reminder we note that $Geom/Geom/1$ system is seen as the discrete time equivalent of $M/M/1$.

As we mentioned earlier ρ is the probability that a cell goes into service given that the server is free (which basically can be interpreted as the fact that either a cell completed its service in the previous time slot or a cell found the queue, and therefore the server, empty upon its arrival). Then the number of time slots between the last departure and the next arrival into service is a (modified) geometric with parameter ρ and probability mass function: $f(k) = (1-\rho)^k\,\rho,\ \ k \geq 0$.

From Figure 3, where we describe the sequence of two successive arrivals and departures with the mean values indicating the various intervals, we can conclude that the following relationship is true in equilibrium:

$$\frac{1}{\lambda} = \frac{1-\rho}{\rho} + \frac{1}{q} \tag{B.17}$$

From the same figure we can also reach the following equation

$$\frac{1}{\lambda} = p_0\frac{1}{\lambda} + \frac{1}{q} \tag{B.18}$$

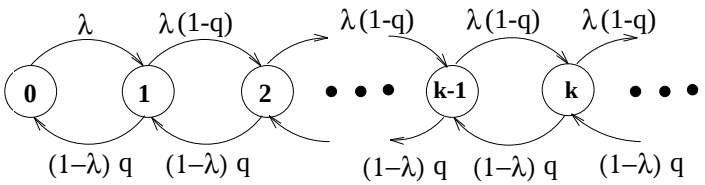

Figure 2: State-transition-probability diagram for the discrete-time Geom/Geom/1 Markov chain.

$$p_k = \frac{\lambda(q-\lambda)}{q^2(1-\lambda)} \ (\frac{\lambda(1-q)}{q(1-\lambda)})^{k-1}, \qquad k \geq 1 \tag{B.13}$$

If $P(z)$ is the z-transform of the number in system distribution then from Equation (B.12) we have

$$P(z) = \sum_{k=0}^{\infty} p_k z^k = (1 - \frac{q}{\lambda}) + (\frac{\lambda(q-\lambda)}{q^2(1-\lambda)} \ \sum_{k=1}^{\infty} (\frac{\lambda(1-q)}{q(1-\lambda)})^{k-1} = \frac{(q-\lambda)(1-\lambda+\lambda z)}{q(1-\lambda)-\lambda z(1-q)} \tag{B.14}$$

As a third way, we mention that $P(z)$ can be found by substituing for $B(z)$ in Equation (A.6). The mean number in system, $\overline{N}$ is then

$$\overline{N} = N'(1) = \frac{\lambda(1-\lambda)}{q-\lambda} \tag{B.15}$$

As a final note we give an expression that relates λ, q and ρ the probability that a cell moves into service given that the server is free (at the beginning of a time slot). We would like first to derive the interdeparture distribution, that is the distribution of the time elapsed (in time slots) between two sucessive cell departures. Let then $E(z)$ denote the z-transform of the interdeparture distribution. Following a HOL cell departure there can be two possible states: either the queue is empty or another cell (already queued) moves into service. Then we can express $E(z)$ as follows:

$$\begin{aligned} E(z) = E(z)|_{\text{queue is nonempty}} \, \mathbf{Pr}[\text{queue is nonempty}] + \\ E(z)|_{\text{queue is empty}} \, \mathbf{Pr}[\text{queue is empty}] \end{aligned} \tag{B.16}$$

$E(z)|_{\text{queue is nonempty}}$ is simply the same as the service time distribution z-transform, namely $B(z)$, while $E(z)|_{\text{queue is empty}}$ is, due again to the memoryless property of the geometric - in this case of $A(z)$- the product of the interarrival time and service time

which of course checks with Equation (B.8).

As an additional exercise we will apply the generalization of the Pollaczek-Khinchin transform equation for a $G/G/1$ system in the (discrete) case of a $Geom/Geom/1$ system [8]. We have

$$H(z) = \frac{\alpha_0[1 - I(\frac{1}{z})]}{1 - C(z)} \tag{B.11}$$

where $\alpha_0 = \mathbf{Pr}$[a cell arrival finds the system idle] and which is equal to $1 - r = 1 - \frac{\lambda}{q}, C(z) = A(\frac{1}{z})B(z)$ and $I(z)$ is the z-transform of the idle period (recall that the latter is a $Geom(\lambda)$). Then Equation (B.11) yields

$$\begin{aligned} H(z) &= \frac{\alpha_0[1 - \frac{\frac{\lambda}{z}}{1-\frac{1-\lambda}{z}}]}{\frac{(z-1)(z-\frac{1-\lambda}{1-q})}{(z-(1-\lambda))(z-\frac{1}{1-q})}} \\ &= \alpha_0 \ \frac{z - \frac{1}{1-q}}{z - \frac{1-\lambda}{1-q}} \end{aligned}$$

which is exactly Equation (B.5) since for this system $\alpha_0 = K$.

As an alternative, we can describe a $Geom/Geom/1$ queueing system as discrete-time. homogeneous Markov chain, which we show in Figure (2) with λ and q as its parameters. Considering the flow conservation accross the boundary between two adjacent states we can establish the following difference equations for the $Geom(\lambda)/Geom(q)/1$ queueing system:

$$\begin{aligned} p_1 &= \frac{\lambda}{q(1-\lambda)}, \\ p_k &= \frac{\lambda(1-q)}{q(1-\lambda)} \ p_{k-1}, \qquad\qquad k \geq 1 \end{aligned}$$

where $p_k = \mathbf{Pr}$[there k cells in the system]. Since now it must be $\sum_{k=0}^{\infty} p_k = 1$ then the single constant p_0 is: $p_0 = 1 - \frac{\lambda}{q}$. Note that we could had directly found p_0 since it is the probability the system is empty thus: $p_0 = 1 - r = 1 - \frac{\lambda}{q}$. It readily follows that:

$$p_0 = 1 - \frac{\lambda}{q} \tag{B.12}$$

$$h_n = (1 - \frac{\lambda}{q})\ u_n + (1 - \frac{\lambda}{q})\ \frac{\lambda}{1-\lambda}(\frac{1-q}{1-\lambda})^n, \qquad \text{n=0,1,2,...} \tag{B.7}$$

where

$$u_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Note that $\overline{H} = H'(1)$ gives the mean waiting time for the $Geom(\lambda)/Geom(q)/1$ system, which yields

$$\overline{H} = \frac{\lambda(1-q)}{q(q-\lambda)} \tag{B.8}$$

then the mean delay can be found to be

$$T = \overline{H} + \frac{1}{q} = \frac{1-\lambda}{q-\lambda} \tag{B.9}$$

We can arrive at Equation (B.9) by considering the mean waiting time expression [8] for a $G/G/1$ system, which we verify here

$$\overline{H} = \frac{\sigma_a^2 + \sigma_b^2 + (\overline{t})^2(1-r)^2}{2\ \overline{t}(1-r)} - \frac{\overline{I^2}}{2\overline{I}} \tag{B.10}$$

where σ_a^2, σ_b^2 are the variance of the interarrival time and service time distributions, $\overline{t}$ is the mean interarrival time, r is the traffic instensity and $\overline{I}$, $\overline{I^2}$ are the first and second moments of the idle distribution. The idle period in our $Geom/Geom/1$ queue starts since the last cell departure and terminates upon the arrival of the next cell. It is important to emphasize that we adopt the convention that in order for an idle period to exist it must be at least one time slot long. Now, since also the geometric distribution enjoys the memoryless property then the time until the next cell arrival is distributed as geometric too, with probability mass function $i(k) = (1-\lambda)^{k-1}\lambda, \quad k = 1, 2, ..,$ which gives the distribution of the idle period (provided there is one).

So, we have $\sigma_a^2 = \frac{1-\lambda}{\lambda^2}, \sigma_b^2 = \frac{1-q}{q^2}, \overline{t} = \frac{1}{\lambda}, r = \frac{\lambda}{q}, \overline{I} = \frac{1}{\lambda}$ and $\overline{I^2} = \frac{2-\lambda}{\lambda^2}$. Therefore Equation (B.10) gives

$$\overline{H} = \frac{\frac{1-\lambda}{\lambda^2} + \frac{1-q}{q^2} + (\frac{1}{\lambda})^2(1-\frac{\lambda}{q})^2}{2\frac{1}{\lambda}(1-\frac{\lambda}{q})} - \frac{\frac{2-\lambda}{\lambda^2}}{2\frac{1}{\lambda}} = \frac{\lambda}{q}\ (\frac{1-q}{q-\lambda})$$

Let λ and q be the parameters of the arrival and service (geometric) probability functions respectively. Then their corresponding probability mass functions are

$$\begin{aligned} a(k) &= (1-\lambda)^{k-1}\lambda, && k = 1, 2, ... \\ b(k) &= (1-q)^{k-1}q, && k = 1, 2, ... \end{aligned}$$

We have immediately that the z-transforms of the two probability mass functions are

$$A(z) = \frac{\lambda z}{1-(1-\lambda)z} \tag{B.1}$$

$$B(z) = \frac{qz}{1-(1-q)z} \tag{B.2}$$

We now proceed with the discrete case version of the *spectrum factorization* [8]. We consider the term $A(\frac{1}{z})B(z)-1$ in order to form a suitable representation of the spectrum factorization where

$$A(\frac{1}{z})B(z) - 1 = \frac{(1-z)(z-\frac{1-\lambda}{1-q})}{(z-(1-\lambda))(z-\frac{1}{1-q})} \tag{B.3}$$

Performing the spectrum factorization we have

$$A(\frac{1}{z})B(z) - 1 = \frac{\Psi_+(z)}{\Psi_-(z)} \tag{B.4}$$

where $\Psi_+(z)$ should be analytic without any zeroes for $|z| < 1$, while $\Psi_-(z)$ has to be analytic and without any zeroes for $|z| > D,\ D < 1$. From Equations (B.3) and (B.4) we get

$$\Psi_+(z) = (1-z)\ \frac{z-\frac{1-\lambda}{1-q}}{z-\frac{1}{1-q}} \qquad \text{and} \qquad \Psi_-(z) = z-(1-\lambda)$$

Let $K = lim_{z\to 1}\dfrac{\Psi_+(z)}{1-z}$, and $\Phi_+(z) = \dfrac{K}{\Psi_+(z)}$. Then $K = 1 - \dfrac{\lambda}{q}$ and the z-transform of the waiting time distribution, $H(z)$, is given by (see [8], chapter)

$$H(z) = (1-z)\Phi_+(z) = K\ \frac{z-\frac{1}{1-q}}{z-\frac{1-\lambda}{1-q}} \tag{B.5}$$

and finally

$$H(z) = (1-\frac{\lambda}{q})\ (1+\frac{\frac{\lambda}{1-\lambda}}{1-\frac{1-q}{1-\lambda}z}) \tag{B.6}$$

By inversion we have

Then $E[\tilde{v}^2] - E[\tilde{v}] = V^{(2)}(1) = \lambda^2 B^{(2)}(1) = \lambda^2(\overline{S^2} - \overline{S})$ where $\overline{S}$ and $\overline{S^2}$ are the first and second moments of the service time. We also verify that $r = E[\tilde{v}] = V'(1) = \lambda B'(1) = \lambda\overline{S}$. Since now $r = \lambda\overline{S}$, Equation (A.3) becomes

$$E[\tilde{p}] = \lambda\overline{S} + \frac{\lambda^2(\overline{S^2} - \overline{S})}{2(1 - \lambda\overline{S})} \tag{A.5}$$

which gives the mean number of cells in the system. Using Little's result we find that the mean delay, T, is merely

$$T = \overline{S} + \frac{\lambda\overline{S(S-1)}}{2(1 - \lambda\overline{S})} \tag{A.6}$$

Forming the z-transform of Equation (A.1) and taking the limit as $n \to \infty$ we further obtain the Pollaczek-Khinchin formula for the *Geom/G/1* queue, which is

$$P(z) = B(1 - \lambda + \lambda z)\,\frac{(1-r)(1-z)}{B(1-\lambda+\lambda z) - z} \tag{A.7}$$

where $P(z)$ is the z-transform of the distribution of the number of cells in the system. According to the $M/G/1$ system we can determine that $P(z) = R(1 - \lambda + \lambda z)$ where $R(z)$ is the z-transform of the total delay distribution in the $Geom/G/1$ queue. Also since $R(z) = H(z)B(z)$ where $H(z)$ is the z-transforms of the waiting time distribution, the latter is given by

$$H(z) = \frac{(1-r)(1-z)}{1 - \lambda - z + \lambda B(z)} \tag{A.8}$$

which is the Pollaczek-Khinchin transform equation [8] for the waiting time in $Geom/G/1$.

Appendix B: The Geom/Geom/1 Queue

We solve for the $Geom/Geom/1$ queue by adopting an approach similar to that of the $G/G/1$ queue [8], where the fundamental consideration is that our system is a discrete-time one (time is divided up into time slots). Events (e.g., arrivals, departures) occur at the boundaries of a time slot.

where p_n is the number of cells left behind in the queue following a cell's departure from service,

$$\Delta_k = \begin{cases} 1 & k = 1, 2, ... \\ 0 & k \leq 0 \end{cases}$$

and v_n are the number of cell arrivals during a service time. Taking expectations on both

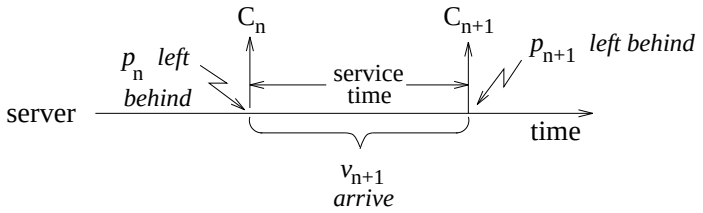

Figure 1: Imbedded Markov chain for a (late arrival) $Geom/G/1$ system.

sides of (A.1) and then taking the limit as $n \to \infty$, we have

$$E[\Delta_{\tilde{p}}] = E[\tilde{v}] = r \tag{A.2}$$

since $E[\Delta_{\tilde{p}}] = \mathbf{Pr}[\tilde{p} > 0] = \mathbf{Pr}[\text{system is busy}]$ where the latter is equal to r, the traffic intensity. As a further step, by squaring Equation (A.1), then taking expectations we can have, at steady-state,

$$E[\tilde{p}] = V^{(1)}(1) - \frac{V^{(2)}(1)}{2(1 - V^{(1)}(1))} = E[\tilde{v}] + \frac{E[\tilde{v}^2] - E[\tilde{v}]}{2(1 - E[\tilde{v}])} = r + \frac{E[\tilde{v}^2] - E[\tilde{v}]}{2(1 - r)} \tag{A.3}$$

where $V(z)$ is the z-transform of the cell arrivals during a service time then clearly

$$V(z) = E[z^{\tilde{v}}] = \sum_{k=0}^{\infty} \mathbf{Pr}[\tilde{v} = k] z^k$$

We have that

$$\mathbf{Pr}[\tilde{v} = k] = \sum_{i=0}^{\infty} \mathbf{Pr}[\tilde{v} = k | \tilde{x} = i] b(i) = \sum_{i=k}^{\infty} \binom{i}{k} \lambda^k (1 - \lambda)^{i-k} b(i)$$

with $b(i)$ the probability mass function of the service time with $B(z)$ as its z-transform. Then $V(z)$ becomes

$$\begin{aligned} V(z) &= \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \binom{i}{k} \lambda^k (1 - \lambda)^{i-k} b(i) z^k = \sum_{i=0}^{\infty} b(i) \sum_{k=0}^{i} \binom{i}{k} (\lambda z)^k (1 - \lambda)^{i-k} = \\ &= \sum_{i=0}^{\infty} b(i)(1 - \lambda + \lambda z)^i = B(1 - \lambda + \lambda z) \end{aligned} \tag{A.4}$$

Rearranging Equation (53) we have

$$\gamma = \frac{M}{N}\lambda q = \alpha\lambda q \tag{54}$$

where $\alpha = \frac{M}{N}$ can be considered as a throughput adjustment factor; where for $\alpha < 1$ it is $\gamma < 1 - (1 - \frac{\lambda}{N})^N$ while for $\alpha > 1$, $\gamma > 1 - (1 - \frac{\lambda}{N})^N$.

For $M = N$, γ is simply

$$\gamma = \mathbf{Pr}[\text{an input is successfull in switching a cell}]$$

Hence

$$q = \mathbf{Pr}[\text{Success}] = \frac{\gamma}{\lambda} = \frac{1 - e^{-\lambda}}{\lambda}$$

4 Conclusion

We presented a complete and simplified analysis of an input-queueing ATM switch where we utilized some key results from queueing theory. As a result, the switch saturation throughput and an exact expression for the mean waiting time are obtained. The study of *Geom/Geom/1* queue led us to deriving explicit expressions for the waiting time and queue length distributions as functions of the input load, namely λ. We also investigated the pure-loss switch for which we obtained its output port occupancy. Based on the latter we rederived the throughput for an $M \times N$ switch.

Appendix A: The Geom/G/1 Queue

Here we examine the $Geom/G/1$ (G discrete) queue by following an approach similar to the study of the $M/G/1$ queue [8]. Since time is slotted, time slots determine the imbedded points in the time axis (we assume here that a departure takes place at the beginning of a time slot). If p_n is the number of cells left behind then from Figure 1 and assuming a late arrival model, we have

$$p_{n+1} = p_n - \Delta_{p_n} + v_{n+1} \tag{A.1}$$

which yields (see Appendix C) :

$$\gamma = 1 - (1 - \frac{\lambda}{N})^M, \qquad 0 \leq \lambda \leq 1 \tag{49}$$

which is exactly Equation (34)

Next, we describe and formaly prove the *flow conservation* relationship for this type of input-queueing system. The flow conservation relationship can be expressed as follows : if q is the probability that a cell is served, provided there is a cell at the input queue, and λ is, again, the probability there is a cell at that input queue then

$$N\gamma = M\lambda q, \tag{50}$$

where γ is the service rate (throughput) of the system. In proving Equation (50) we have

$$\begin{aligned} q &= \mathbf{Pr}[\text{Success}] \\ &= \mathbf{Pr}[\text{A cell wins the output contention}] \\ &= \mathbf{Pr}[\text{A cell is switched from an input}|\text{there is a cell at the input}] \end{aligned}$$

The conditional probability is then

$$q_{|\text{k cells are switched}} = \sum_{i=0}^{M-1} \left(\begin{array}{c} N \\ k \end{array} \right) \left(\begin{array}{c} M-1 \\ i \end{array} \right) \lambda^{M-1-i}(1-\lambda)^i \frac{\beta_{i,k}}{N^{M-i}}, \quad 0 \leq k \leq N \tag{51}$$

where $M-1$ accounts for the fact: it is given that there exists at least one (out of possible M) cell at an input port and the $M-1-i$ cells ($i = 0, ..., M-1$) are present with probability λ each. From Equation (51) we can form the unconditional probability

$$\begin{aligned} q &= \sum_{k=1}^{N} \sum_{i=0}^{M-1} \left(\begin{array}{c} N \\ k \end{array} \right) \left(\begin{array}{c} M-1 \\ i \end{array} \right) \lambda^{M-1-i}(1-\lambda)^i \frac{\beta_{i,k}}{N^{M-i}} \frac{k}{M-i} \\ &= \sum_{k=1}^{N} \sum_{i=0}^{M-1} \left(\begin{array}{c} N \\ k \end{array} \right) \left(\begin{array}{c} M-1 \\ i \end{array} \right) \frac{N}{M-i} \lambda^{M-1-i}(1-\lambda)^i \frac{\beta_{i,k}}{N^{M-i}} \frac{k}{N} \\ &= \frac{N}{\lambda M} \sum_{k=1}^{N} \sum_{i=0}^{M} \left(\begin{array}{c} N \\ k \end{array} \right) \left(\begin{array}{c} M \\ i \end{array} \right) \lambda^{M-i}(1-\lambda)^i \frac{k\beta_{i,k}}{N^{M+1-i}} \end{aligned} \tag{52}$$

where we recognize that the double sum in Equation (52) is simply the mean throughput (see Equation (48)). Thus we immediately obtain

$$q = \frac{N\gamma}{\lambda M} \qquad \text{or} \qquad N\gamma = M\lambda q \tag{53}$$

A few observations regarding Equation (39): $\beta_{0,k} = m_k$, for $0 \leq k \leq N$ and also, it turns out that, in fact, $\beta_{i,k} = 0$ for $i > M - k,\ k \leq\ min\{M, N\}$ and for any $k > min\{M, N\}$ [15].

Note also that, for $M = N$, $d^*(N)$ can be simply expressed as

$$d^*(N) = \frac{N!}{N^N} \tag{42}$$

and taking the limit of Equation (42) as $N \to \infty$, for $\lambda = 1$ we can conclude that

$$\lim_{N \to \infty} d(N) = 0 \tag{43}$$

which can be interpreted as the probability of all outputs being busy is very close to 0 for a very large N (and under heavy traffic). This is generally true also for any $M > N$.

We can easily verify that

$$\sum_{k=1}^{N} d^*(k) = 1 \tag{44}$$

From Equation (38) we immediately see that for $\lambda = 1$, $\quad d(k) = d^*(k)$, so the second term in the right-hand side of Equation (38) accounts for $\lambda < 1$. We can furthermore check (see Appendix C) that

$$\sum_{k=0}^{N} d(k) = 1 \tag{45}$$

According to Equations (38) and (39), $d(N)$ is given by

$$d(N) = d^*(N)\lambda^M \tag{46}$$

Taking Equation (40) into account then Equation (38) can be rewritten as follows:

$$d(k) = \left(\begin{array}{c} N \\ k \end{array} \right) \sum_{i=0}^{M} \left(\begin{array}{c} M \\ i \end{array} \right) \lambda^{M-i}(1-\lambda)^i \frac{\beta_{i,k}}{N^{M-i}}, \qquad 0 \leq k \leq N \tag{47}$$

Proceeding in obtaining the throughput, by applying (47) in (37), we get

$$\gamma = \sum_{k=0}^{N} \sum_{i=0}^{M} \left(\begin{array}{c} N \\ k \end{array} \right) \left(\begin{array}{c} M \\ i \end{array} \right) \lambda^{M-i}(1-\lambda)^i \frac{k\beta_{i,k}}{N^{M+1-i}}, \tag{48}$$

[15] This is consistent since $M - i \geq k$ thus $i \leq M - k$ and, $d(k) = 0$ for any $k > M$.

send) cells get switched (the remaining $M-k$ are dropped),

$$d(k) = \mathbf{Pr}[\text{exactly k cells are in service }^{14}] \tag{36}$$

Then the throughput of the system is given by forming the expectation

$$\gamma = \sum_{k=0}^{N} d(k)\frac{k}{N} \tag{37}$$

where $\frac{k}{N}$ is the instantaneous throughput, if only k cells were switched during a time slot.

Recalling that λ is the probability that there is a cell in an input (single-cell) buffer, then $d(k)$ is described by

$$\begin{aligned} d(0) &= (1-\lambda)^M \\ d(k) &= d^*(k)\lambda^M + \sum_{i=1}^{M-1} \left(\begin{array}{c} N \\ k \end{array} \right) \left(\begin{array}{c} M \\ i \end{array} \right) \lambda^{M-i}(1-\lambda)^i \frac{\beta_{i,k}}{N^{M-i}}, \qquad 1 \le k \le N \end{aligned} \tag{38}$$

where $\beta_{i,k}$ is given by the following recursive formula:

$$\begin{aligned} \beta_{i,0} &= \left\{ \begin{array}{ll} 0, & 0 \le i \le M-1 \\ 1, & i = M \end{array} \right. \\ \beta_{i,k} &= k^{M-i} - \sum_{j=0}^{k-1} \left(\begin{array}{c} k \\ j \end{array} \right) \beta_{i,j}, \qquad 0 \le i \le M, \quad 1 < k \le N \end{aligned} \tag{39}$$

while $d^*(k)$ gives the occupancy distribution for $\lambda = 1$, namely at saturation (all single input buffers have cells)

$$d^*(k) = \frac{\left(\begin{array}{c} N \\ k \end{array} \right) m_k}{N^M}, \qquad 1 \le k \le N \tag{40}$$

where

$$\begin{aligned} m_0 &= 0 \\ m_k &= k^M - \sum_{j=0}^{k-1} \left(\begin{array}{c} k \\ j \end{array} \right) m_j, \qquad 1 \le k \le N \end{aligned} \tag{41}$$

[14] In the process of being transmitted to the output trunks.

We consider an $M \times N$ switch where generally[13] $M \geq N$. Cell arrivals to the input trunks are, as before, i.i.d. according to a Bernoulli process with parameter λ. Arrivals are assumed to occur at the beginning of the time slot and the output ports are equally favored as destinations for the incoming cells. At the end of the time slot arbitration for the output trunks takes place and if there are, i.e., c ($c \geq 1$) cells destined to a particular output then one of them is randomly chosen by the switch arbitrator to access the output port, while the remaining $c-1$ cells are dropped, thus lost, from the switch.

It is apparent that this pure-loss system is suitable for heavy-load input traffic situations, namely for a sufficiently large λ. The throughput for this type of switching system and for the general case of an $M \times N$ switch, has been derived [11] and is given by

$$\gamma = 1 - (1 - \frac{\lambda}{N})^M \tag{34}$$

As $M, N \to \infty$, then

$$\gamma = 1 - e^{-\frac{M}{N}\lambda} \tag{35}$$

where for $\lambda = 1$ (each and every buffer has a cell to transmit) and $M = N$, γ becomes $\gamma = 1 - e^{-1} \approx 0.632$, which is simply the probability that an input successfully switches a cell to the appropriate output.

We now find the exact distribution of the number of output ports (servers) that are busy (received a cell from an input) and based on that occupancy distribution we easily rederive Equation (35).

Our approach is combinatorial and is primarily based on enumerating the possible different cases of cells buffered at the inputs, given that only k, $0 \leq k \leq N$, of them are addressed to the different outputs (similar to the birthday problem). We study the general case, that is of an $M \times N$ switch. Let then $d(k)$ denote the probability that exactly k outputs become busy during a typical time slot, that is exactly k out of M (in case all inputs have cells to

[13]Our approach holds for any relation between M and N.

2.4 The General Case of an $M \times N$ Switch

Here we consider an $M \times N$ switch, where M and N are the number of input and output ports respectively and also let $M = \alpha N$, where α is a positive integer. Then if γ is the throughput for this kind of switch and λ is the cell arrival rate into each of the M input queues then $N\gamma = M\lambda = \alpha N\lambda$ from which we get $\gamma = \lambda\alpha$. Equation (7) holds also here with $\lambda = \frac{\gamma}{\alpha}$. Thus Equation (10) can be written as follows:

$$A_j(z) = (1 - \frac{\gamma(1-z)}{\alpha N})^M = (1 - \frac{\gamma(1-z)}{M})^M \tag{31}$$

and as $M \to \infty$, A_j becomes a Poisson Process with parameter γ. Accordingly,

$$q = \frac{2(1-\gamma)}{2-\gamma} \tag{32}$$

and since at saturation $q = \lambda = \frac{\gamma_{max}}{\alpha}$ then we find that

$$\gamma_{max} = 1 + \alpha - \sqrt{1+\alpha^2} \tag{33}$$

which represents the maximum throughput of an $\alpha N \times N$ switch. Note that Equation (33) holds for any value of N, i.e., for $N < \infty$, which then means that α is very large, thus $\gamma_{max} \to 1$. If now $M < \infty$ then A_j is still Poisson, and Equation (33) still applies; then $\gamma_{max} \to 0$, since now α is very small. Again, γ is a measure of the switch utilization (i.e. fraction of time output ports are transmitting cells). It is then obvious that in the case where $M < \infty, N \to \infty$ there is no HOL blocking and all new incoming cells get switched immediately. It should be mentioned that, in general, cases where $M < \infty$ and $N \to \infty$ or $M \to \infty$ and $N < \infty$ are rather impractical in building an ATM switch.

3 The Pure-Loss Model and its Output Port Occupancy Distribution

In section 1 we mentioned that an alternative for eliminating any HOL blocking is to consider the *pure-loss* model, where essentially cells are discarded from the switch, if they become blocked after the output port contention.

and by a simple inspection we get

$$s_n = u_{n-1} \circledast \ w_n = w_{n-1}, \qquad n = 1, 2, 3, ... \tag{26}$$

where, by definition,

$$u_n \ = \ \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Equation (26) finally leads to

$$s_n = \frac{\lambda^2 - 4\lambda + 2}{(1-\lambda)(2-\lambda)} \ (\frac{\lambda}{(1-\lambda)(2-\lambda)})^{n-1}, \qquad n = 1, 2, 3, ... \tag{27}$$

2.3 Distribution of Queue Length

In Appendix B we derive the distribution of the number in system for the $Geom(\lambda)/Geom(q)/1$ queueing system. We observe that the number in system is exactly the same as the (input) queue length for our ATM switch. Therefore, we can directly adopt the results derived in Appendix B, namely Equations (B.13)-(B.15) which we rewrite here, after substituing for q (as given in Equation (13)). The queue length probabiliy mass function is

$$\begin{aligned} q_0 \ &= \ \frac{\lambda^2 - 4\lambda + 2}{2(1-\lambda)} \\ q_k \ &= \ \frac{(2-\lambda)(\lambda^2 - 4\lambda + 2)}{2\lambda(1-\lambda)} \ (\frac{\lambda^2}{2(1-\lambda)^2})^k, \qquad k \geq 1 \end{aligned} \tag{28}$$

while the z-transform of the queue length distribution is

$$Q(z) = \frac{(\lambda^2 - 4\lambda + 2)(1 - \lambda + \lambda z)}{2(1-\lambda)^2 - \lambda^2 z} \tag{29}$$

It is easy then to show that the mean queue length, $\overline{Q}$, is

$$\overline{Q} = \frac{\lambda(1-\lambda)(2-\lambda)}{\lambda^2 - 4\lambda + 2} \tag{30}$$

As we would anticipate, this last expression can be obtained directly from Equation (25) and Little's result. Notice that **Pr**[no wait in the input-queue] $= w_0 > q_0 =$**Pr**[input-queue is empty] which is of course insightful since a cell get switched immediately upon its arrival to an empty queue.

time distribution, $H(z)$, for the $Geom(\lambda)/Geom(q)/1$ queue[12]. If $B(z)$ is the z-transform of the service time distribution, for the $Geom/Geom/1$ queueing system, then we can write

$$zW(z) = H(z)B(z) \tag{21}$$

Since (see Appendix B)

$$B(z) = \frac{qz}{1 - 1(1-q)z} \qquad \text{and} \qquad H(z) = (1 - \frac{\lambda}{q}) \ \frac{z - \frac{1}{1-q}}{z - \frac{1-\lambda}{1-q}}.$$

Equation (21) yields

$$W(z) = \frac{q - \lambda}{1 - \lambda - (1-q)z} = \frac{q - \lambda}{1 - \lambda} \ \frac{1}{1 - \frac{1-q}{1-\lambda}z}$$

where q is given in Equation (13). Hence

$$W(z) = \frac{\lambda^2 - 4\lambda + 2}{\lambda^2 - 3\lambda + 2 - \lambda z} \tag{22}$$

This last may be inverted and finally yield

$$w_n = \frac{\lambda^2 - 4\lambda + 2}{(1-\lambda)(2-\lambda)} \ (\frac{\lambda}{(1-\lambda)(2-\lambda)})^n, \qquad n = 0, 1, 2, .. \tag{23}$$

We see that the waiting time density is of geometric form which should not surprise us. From Equation (22) we get the mean waiting time (as an approximation)

$$\overline{W} = W' = \frac{\lambda}{\lambda^2 - 4\lambda + 2} \tag{24}$$

which can be also derived from Equation (B.9) after substituting for q (Equation (13)). The T, the mean delay is

$$T = \overline{W} + 1 = \frac{(1-\lambda)(2-\lambda)}{\lambda^2 - 4\lambda + 2} \tag{25}$$

The reader can compare expressions (24) and (25) to (19) and (18) respectively.

The z-transform, $S(z)$, of the total delay distribution, s_n, for the switch, is

$$S(z) = zW(z)$$

[12] $W(z)$ is different than $H(z)$; recall that service in the $Geom/Geom/1$ queue reflects the HOL holding plus switching time.

Then Equation (16) can be written as

$$\overline{W_{ROS}^2} = \frac{\lambda(\lambda+2)}{3(2-\lambda)(1-\lambda)^2}$$

Thus

$$\overline{S^2} = \overline{S_{ROS}^2} = \overline{W_{ROS}^2} + 2\,\overline{W}_{M/D/1} + 1 = \frac{4\lambda^2 - 7\lambda + 6}{3(1-\lambda)^2(2-\lambda)} \tag{17}$$

From Equations (2), (13) and (17) we finally have that

$$T = \frac{6\lambda^4 - 31\lambda^3 + 76\lambda^2 - 72\lambda + 24}{6(2-\lambda)(1-\lambda)(\lambda^2 - 4\lambda + 2)} \tag{18}$$

while the mean waiting time is simply

$$\overline{W} = T - 1 = \frac{11\lambda^3 - 20\lambda^2 + 12\lambda}{6(2-\lambda)(1-\lambda)(\lambda^2 - 4\lambda + 2)} \tag{19}$$

As a last remark we add that: if N_j denotes the number of cells in the virtual queue V_j (i.e., HOL cells requesting the same output j), then from Little's result we have $E[N_j] = \lambda\overline{S} = \dfrac{\lambda}{q}$ which implies (from Equation (13))

$$E[N_j] = \lambda + \frac{\lambda^2}{2(1-\lambda)} \tag{20}$$

Also from Equation (3) we have that $E[N_j] = \dfrac{\lambda}{q} = 1 - p_0 = r$. At saturation, since $\lambda = q$, it is $E[N_j] = 1$.

2.2 Distribution of Waiting Time

We mentioned that q represents the probability that a HOL cell is routed during a time slot (which of course also corresponds to the probability that a HOL cell wins the output contention). Then an input-queue can be approximated by a $Geom(\lambda)/Geom(q)/1$ queue. Utilizing this model we want to find the distribution of the total time spent by a cell in the switch (until it is completely switched out). Let $W(z)$ be the z-transform of the waiting time distribution w_n for the switch. In Appendix B we derive the z-transform of the waiting

Since the service time, S, of the $Geom(\lambda)/G/1$ queue, is equal to the delay of the $M(\lambda)/D/1$ queue with random oreder of service (ROS), we have

$$\overline{S} = \overline{W}_{M/D/1} = \frac{2-\lambda}{2(1-\lambda)} \tag{12}$$

From Equations (1) and (12) we finally get

$$q = \frac{2(1-\lambda)}{2-\lambda} \tag{13}$$

Voilá. We now have an expression for q, which is, as we promissed, a function of λ. Equating now q, as given in (13) to λ, and then solving for λ gives the well-known maximum attainable throughput, γ_{max}, for the switch

$$\gamma_{max} = 2 - \sqrt{2} \approx 0.586 \tag{14}$$

Taking ρ into consideration we can alternatively obtain the following quadratic equation (by substituting $\overline{S}$ in Equation (4):

$$(2-\rho)\lambda^2 - 2(1+\rho)\lambda + 2\rho = 0 \tag{15}$$

where solving for λ at $\rho = 1$ gives Equation (14) [11].

Proceeding now with finding the mean cell (switch transit) delay, as expressed in Equation (2) we have that the second moment of the waiting time distribution in a $M(\lambda)/D/1$ queueing system is given (see [12] pg.)

$$\overline{W^2_{ROS}} = \frac{1}{1-\frac{\lambda}{2}}\ \overline{W^2_{FCFS}} \tag{16}$$

where $\overline{W^2_{FCFS}}$ is the second moment of the waiting time distribution of an $M/G/1$ queue with FCFS service. The latter for an $M(\lambda)/D/1$ system is

$$\overline{W^2_{FCFS}} = \frac{\lambda(\lambda+2)}{6(1-\lambda)^2}$$

[11] Note that $\rho = 1$ corresponds to a saturation situation where $\lambda = q$, and $\rho = 1$ iff $r = 1$. Thus $\rho > 1$ or $r > 1$ leads to an unstable condition for the switching system.

unblocked] then u is given by

$$u = p_0 + (1 - p_0)q \tag{8}$$

From Equations (7) and (8) we immediately notice that $u = \frac{\lambda}{\rho}$. Thus we have the following binomial probabilities :

$$v_h = \left(\begin{array}{c} N \\ h \end{array} \right) u^h (1-u)^{N-h} = \left(\begin{array}{c} N \\ h \end{array} \right) (\frac{\lambda}{\rho})^h (1 - \frac{\lambda}{\rho})^{N-h}, \qquad h = 0, 1, 2, ..., N \tag{9}$$

Equation (6) then yields[9]

$$\begin{aligned} A_j(z) &= \sum_{h=0}^{N} (1 - \frac{\rho(1-z)}{N})^h \left(\begin{array}{c} N \\ h \end{array} \right) (\frac{\lambda}{\rho})^h (1 - \frac{\lambda}{\rho})^{N-h} \\ &= (1 - \frac{\lambda(1-z)}{N})^N \end{aligned} \tag{10}$$

which is independent of ρ.

We can further conclude that as $N \to \infty$ then the probability mass function of A_j, the number of *fresh* HOL cells, becomes a Poisson Process with parameter λ. As a side comment we mention that from the distribution of the number of unblocked inputs given in Equation (9) we find that the expected number of free inputs is

$$E[H] = \sum_{h=0}^{N} h v_h = N \frac{\lambda}{\rho}$$

which proves the *second flow conservation* relationship [3]

$$E[H]\rho = N\lambda \tag{11}$$

that describes the conservation of flow across the entire switching system.

In what follows we assume[10] that $N \to \infty$. According to our discussion above, the virtual queue V_j is then modelled as an $M(\lambda)/D/1$ queue with random service. It follows from the $M/G/1$ theory [8] that the mean delay for this system is merely

$$\overline{W}_{M/D/1} + 1 = \frac{\lambda}{2(1-\lambda)} + 1 = \frac{2 - \lambda}{2(1-\lambda)}$$

[9] We could have arrived at the same equation in a single step if we had considered the **Pr**[a fresh cell is destined to the j-th output] $= \frac{\lambda}{\rho}\frac{\rho}{N} = \frac{\lambda}{N}$.

[10] In actuality, it is $N \geq 20$, since we know as a rule of thumb that it is acceptable to use the Poisson approximation for a binomial distribution for $N \geq 20$. In fact we can verify that by looking at the simulation figure 2 in [6] where the throughput curve remains almost flat as the switch size, N, grows more than 20.

are H unblocked input queues at the beginning of a typical time slot (either they had no cells in the previous slots or a HOL cell was just released), then for a given H, i.e., $H = h$, we can write :

$$\begin{aligned} A_{j|H=h}(z) &= E[z^{A_j}|H=h] = \sum_{k=0}^{h} \mathbf{Pr}[A_j = k] z^k = \sum_{k=0}^{h} \left(\begin{array}{c} h \\ k \end{array} \right) (\frac{\rho}{N})^k (1 - \frac{\rho}{N})^{h-k} z^k \\ &= (1 - \frac{\rho(1-z)}{N})^h \end{aligned} \tag{5}$$

since a fresh cell destined to the j-th output can appear at the HOL position with probability $\frac{\rho}{N}$. Let us assume for the moment that N is a very large number [7]. Removing now the condition from Equation (5) we get

$$A_j(z) = \sum_{h=0}^{N} A_{j|H=h}(z) \mathbf{Pr}[H = h] \tag{6}$$

Let us calculate $v_h = \mathbf{Pr}[H = h]$, the steady-state probability that there are h ($h \leq N$) unblocked queues (or free HOL positions). Note that ρ also describes the rate that cells are moving into the HOL position. Therefore, at steady-state we have :

$$(1 - p_0) q \rho + p_0 \rho = \lambda \tag{7}$$

where the first term in the right-hand side, namely $(1-p_0)q\rho$, gives the rate cells are moving into the HOL position following the departure of a HOL cell, where the departure does not leave an empty input queue behind it and the second term, $p_0\rho$, refers to the rate incoming cells move into the HOL position, assuming they find an empty input queue upon their arrival and λ is, again, the input arrival rate[8]. If we now define $u = \mathbf{Pr}$[input queue is

[7] We later refer to which large numbers are acceptable.

[8] A more formal proof of Equation (7) can be obtained if we consider Equation (4) and by taking note of the flow conservation, Equation (3). Another rigorous proof based on basic probability principles can be drawn as follows: let A' denote the event of a fresh HOL cell, A the event where an input queue was blocked in the previous slot, B the event that a HOL departed in the previous slot and C the event the input queue was empty in the previous slot. Note that from our definition of events A, B and C are mutually exclusive ($B \bigcap C = \emptyset$) and that $\mathbf{Pr}[A] = (1-q)(1-p_0)$, $\mathbf{Pr}[B] = q(1-p_0)$ and $\mathbf{Pr}[C] = p_0$ and $\mathbf{Pr}[A \bigcup B \bigcup C] = 1$. Then

$$\rho = \mathbf{Pr}[A'|(B \bigcup C)] = \frac{\mathbf{Pr}[A' \bigcap (B \bigcup C)]}{\mathbf{Pr}[B \bigcup C]} = \frac{\mathbf{Pr}[A' \bigcap B] + \mathbf{Pr}[A' \bigcap C]}{\mathbf{Pr}[B] + \mathbf{Pr}[C]} = \frac{\lambda}{q(1-p_0) + p_0}$$

since the throughput is equal to the arrival rate into the HOL position, thus $\lambda = \mathbf{Pr}[A'] = \mathbf{Pr}[A' \bigcap (A \bigcup B \bigcup C)] = \mathbf{Pr}[A' \bigcap A] + \mathbf{Pr}[A' \bigcap B] + \mathbf{Pr}[A' \bigcap C] = \mathbf{Pr}[A' \bigcap B] + \mathbf{Pr}[A' \bigcap C]$ since, obviously, $\mathbf{Pr}[A' \bigcap A] = 0$.

input queues is (see Equation (A.6))

$$T = \overline{S} + \frac{\lambda(\overline{S^2} - \overline{S})}{2(1 - \lambda\overline{S})} = \frac{1}{q} + \frac{\lambda(q\overline{S^2} - 1)}{q - \lambda} \tag{2}$$

Obviously, at saturation $r = 1$ thus $\lambda = q$, in which case the mean delay becomes infinitely large and λ then yields the maximum, saturation, throughput of the switch Yet, it remains to determine q which will enable us to apply the results from the $Geom/Geom/1$ queueing analysis. We embark upon finding q by focusing on the virtual queue formed by those cells contending for the same "tagged" output j. We refer to this virtual queue as V_j. It is assumed that the service order in V_j is random (the winner HOL cell is selected randomly among all cells in V_j). Other selection disciplines include a FCFS (if a round-robin scheduling is assumed, i.e. the longest waiting HOL cell is selected) or a priority-based one (the longest input queue is chosen).

We first give a rather intuitive result which is also described as the *flow conservation* relationship [4]. To the set of assumptions mentioned in the beginning we add that: each HOL cell has the same probability, namely q. of being switched (served) at the beginning of a time slot. We feel that it is fair to make such an assumption given the steady-state of the system. We mentioned that the traffic intensity of an input queue is $r = \frac{\lambda}{q}$; r is also equal to the $\mathbf{Pr}$[an input queue is not empty] thus $r = 1 - p_0$; hence we immediately get

$$\lambda = (1 - p_0) \times q \tag{3}$$

which is the flow rate conservation describing an input queue. Here, we rewrite Equation (B.19), which holds for a $Geom(\lambda)/G/1$ queue,

$$\frac{1}{\lambda} = \frac{1 - \rho}{\rho} + \overline{S} \tag{4}$$

where ρ is the probability that a cell moves into the HOL position, given the HOL is free (unblocked)[6]. We call any new arrivals at the HOL position *fresh* HOL cells.

Let also A_j denote the number of the *fresh* HOL cells destined to a particular output j. We want to show that the distribution of A_j is Poisson as $N \to \infty$. If we assume that there

[6] We include ρ in our analysis in order for our analysis to be comparable to the results presented in [3].

the interested reader the complete versions of the appendices can be found in [7].

2 Input-buffering: A Purely Queueing Theoretic Approach

2.1 The Asymptotic Throughput and Mean Delay

An input queue in a single-stage, space-division switch is modelled as a time-slotted $Geom/G/1$ queueing system, where a cell arrives at a queue with probability λ which also denotes the arrival rate. We allow cell arrivals to occur prior to the time slot boundary (thusly we consider a *late arrival* model [13]), while departures take place after arrivals (and once the contention phase is over) at the beginning of a time slot. Let S denote the service time with distribution G. Once a cell moves into the HOL position it is assumed to be in service until it is finally switched out from its desired output port[5]. Let denote by $1/q$ the mean service time, $\overline{S}$, in our system. Thus

$$\overline{S} = \frac{1}{q} \tag{1}$$

What is special about this queueing system, as we will see later, is that the mean service time $1/q$ or simply q depends on λ, the input traffic load. It can be thought that q also represents the probability that a queue independently and randomly chosen sends its HOL cell to its desired output. In modelling each input queue as a $Geom(\lambda)/G/1$ queue we make the following fundamental assumptions:

- Input queues in the $N \times N$ switch are independent from each other and they all accept the same traffic load λ (balanced traffic).
- An incoming cell can equiprobably choose any of the N outputs (uniform traffic).

In order for the queue to be stable we require that $r = \lambda\overline{S} = \dfrac{\lambda}{q} < 1$, where r is the traffic intensity. The mean delay that a typical cell will experience upon its arrival at one of the

[5] A cell to find an input queue empty proceeds immediately to the HOL position, thus into service as fas as the $Geom/G/1$ queue is concerned.

input-queue length distribution and mean waiting time, based on an approximate model where the input queues are considered $Geom/Geom/1$ queueing systems[3].

We tackle the problem of switches with input-queueing from a completely different viewpoint. Our analysis is simpler mainly in two aspects: (a) we directly obtain the switch's throughput by considering the mean cell transit delay and (b) we follow a rather concise and straighforward approach in proving that the number of any HOL arrivals is Poisson. Exact results for the switch's maximum throughput and mean delay are derived, where by throughput we mean the utilization of the output ports (fraction of time they are busy, i.e. they have a packet to transmit). Furthermore, we obtain an expression for the queue length distribution (that matches the one in [3]) and the waiting time distribution distribution which constitutes a new result. These latter results are based on using the $Geom/Geom/1$ approach. The throughput of an $M \times N$ switch (M inputs, N outputs) follows, as a generalization to our previous results.

The second part of the paper is focused on studying the case where there is no queueing in the switch (assuming an $M \times N$ switch). Effectively, the switch drops any cells that are blocked (i.e., any that lose the output contention). This model, which we call the *pure-loss* switch, has been applicable in studying processor-memory interconnection networks [11]. We provide an analytical expression for the output port occupancy probability mass function. From the latter we can mathematically derive the throughput for this type of switching architecture[4].

We conclude by summarizing our main results. At the end of the paper there are three appendices. In the first one the reader can find a short but indicative treatment of the $Geom/G/1$ queue while the second one provides a solution to the $Geom/Geom/1$ queue using the spectral method [8]. Both systems are being studied in the context of ATM switching. The third appendix presents the proofs of some equations given in section 3. For

[3]The approximation pertains to the fact that the HOL holding time distribution is assumed to be Geometric. Karol *et al* consider a $Geom/G/1$ model for the input queue [5].

[4]Though a more direct approach can be used (e.g., [11], [4]) to find the switch's maximum throughput, our method serves as a way of validating our results.

The low performance in an input-queueing switching system is attributed to a phenomenon called *head-of-line* (HOL) blocking. HOL blocking is caused when more than one cell tries to access the same output. Assuming that only one of the contending HOL cells can be routed through the requested output port the remaining HOL cells stay blocked[1], thus blocking also the cells (if any) that are queued behind them. In effect, some output ports can be idle during a time slot (time is slotted, where each time slot is equal to the transmission time of a cell).

In the first part of this paper we present a complete analysis of an input-queueing switch under a uniform traffic pattern. Our study establishes an alternative and rather simplified approach to those previously followed in [5], [3]. In [5] Karol *et al* show that the asymptotic saturation throughput of a packet switch (packets are assumed of fixed size) under uniformly i.i.d. Bernoulli input traffic is $2 - \sqrt{2} \approx 0.586$. Their analysis is based on the equation expressing the number of packets destined for a tagged output (in a typical time slot) as a function of the number of packets destined for the same output in the previous slot and of the number of any new HOL arrivals destined also for the same output. Also, in the first appendix of their paper they prove, using the Chebyshev inequality, that, in the limit (assuming an infinite number of ports in the switch), the number of new HOL arrivals destined to the same output is Poisson with the input traffic rate as its parameter. In their paper the mean waiting time can be numerically calculated.

Hui and Arthurs deal with the same problem on input-queueing [3] using basically the same fundamental equation (as in [5]) expressing the number of requests for the tagged output. Instead of proving directly that the number of new HOL arrivals destined for the same output follows a Poisson distribution they obtain the first two moments of their distribution (which, of course, are the same as those of the Poisson). Based on an approximated model where the input queues are considered $Geom/Geom/1$ queueing systems[2] They derive the

[1] We assume that incoming cells are queued in a first-come-first-serve (FCFS) fashion.

[2] The approximation pertains to the fact that the HOL holding time distribution is assumed to be Geometric. Karol *et al* consider a $Geom/G/1$ model for the input queue [5].

Yet another analytic study of input-buffered ATM switches †

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Abstract

We present a new, simpler than previously presented, refined approach in evaluating analytically the performance of an input-buffered ATM switching system under uniform-traffic. In addition to the input-queueing model, we study a switch model where there is no queueing whatsoever (therefore inducing cell-loss inside the switch) and which we call the *pure-loss* switch. In evaluating the former model, we use pure queueing theory tools to derive the system throughput, mean delay, waiting time distribution and queue length distribution. Some well-known results are rederived: the throughput of an input-queueing switch, which is $2 - \sqrt{2} \approx 0.586$ and the throughput of the pure-loss switch, which is $e^{-1} \approx 0.632$.

1 Introduction

Recognizing the importance of ATM switching in ATM networks numerous packet switching architectures have been proposed, modeled, studied and implemented ([1],[10]). The merits and tradeoffs of both *input-queueing* (where cell buffers are placed at the switch's inputs) and *output-queueing* (where buffers are located at the outputs) strategies have been extensively investigated. Briefly, we mention that while input-queueing is simple and less expensive to implement it suffers from a low performance, in terms of throughput and mean (transit) delay. On the other hand, output-queueing switches enjoy a much higher performance at the price of a more complex switching mechanism.

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