Axiomatic Characterization of Directed Graphs

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Abstract

While graphs are normally defined in terms of the 2-place relation of adjacency, we take the 3-place relation of interception as the basic primitive of their definition. building on results previously obtained for undirected graphs, this paper establishes an axiomatic characterization of 3-place relations that lend themselves to representation in terms of path interception in directed graphs, thus providing a new characterization of graphs.

- We will refer to our previous characterization paper from June '93 as [PPU]
- Given a directed graph (not necessarily acyclic) define $R_G(X, Z, Y)$ as "all the directed paths from X to Y are intercepted by Z."

1 Axioms

X.2.1 $R(X, ZY, W) \rightarrow R(X, Z, Y) \wedge R(X, Z, W)$
X.2.2 $R(XW, Z, Y) \rightarrow R(X, Z, Y) \wedge R(W, Z, Y)$
X.3 $R(X, Z, Y) \rightarrow R(X, ZW, Y)$ all W
X.4.1 $R(X, ZW, Y) \wedge R(X, ZY, W) \rightarrow R(X, Z, WY)$
X.4.2 $R(X, ZW, Y) \wedge R(W, XZ, Y) \rightarrow R(XW, Z, Y)$
X.5 $R(X, Z, Y) \rightarrow R(a, Z, Y) \vee R(X, Z, a)$ all $a \notin X \cup Z \cup Y$

X.2 = Decomposition, X.3 = Strong union
X.4 = Intersection, X.5 = Transitivity

Theorem 1.1 *If R is a relation induced by a directed graph then R is closed under all the above axioms.*

Proof: X.2 and X.3 hold trivially.

X.5: If the left hand side of 5 holds then we cannot have a directed path from a to Y together with a directed path from X to a.

X.4.1: Proof is exactly the same as in [PPU, page 4] with "directed path" substituted for "path" and "paths from A to B" substituting for "paths between A and B".

X.4.2: If both X and W are disconnected from Y in G then trivial. If X is disconnected from Y but W is connected into Y (the other case is similar) then $R(W, ZX, Y)$ implies that all directed paths from W to Y are intercepted by Z before entering $Y \Rightarrow R(WX, Z, Y)$.

The remaining case is the case where there are directed paths from X and W into Y.

Assume, by way of contradiction, that $\neg R(XW, Z, Y)$. Then there is a path from X into Y not intercepted by Z or there is a path from W to Y not intercepted by Z. W.l.o.g assume the first alternative. From $R(X, ZW, Y)$ on the l.h.s. of X.4.2 we deduce that such paths (from X into Y not intercepted by Z) must be intercepted by W:

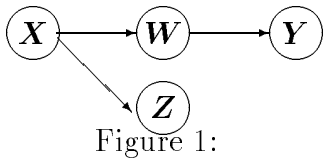

Figure 1:

This implies that there are paths from W to Y not intercepted by X and not intercepted by Z contradicting $R(W, ZX, Y)$ on the l.h.s. of X.4.2. □

Lemma 2 from [PPU] carries over with (6) split into 6.1 (as the original) and 6.2 as below

$$6.2 \quad R(X, Z, Y) \wedge R(W, Z, Y) \rightarrow R(XW, Z, Y)$$

Proof is as in [PPU] now using the bidirectional axioms which make symmetry superfluous.

2 Graph Characterization (DG)

Definition of $G_0(a \rightarrow b) \in E_G$ iff $(a, V/\{a, b\}, b) \notin R$.

Theorem 2.1 *If R is closed under the Axioms then G_0 is a perfect map of R.*

3 Proof of I-Mapness

By induction on the size of the middle set in a triplet.

Basis: If $t = (a, V/\{a, b\}, b)$ then

$$t \in G_0 \Rightarrow (a \rightarrow b) \notin E_G \Rightarrow t \in R$$

Step: Assume $t = (X, Z, Y) \in R_G$ for all Z with $|Z| = k \leq n - 2$ and fixed k.

Let $t' = (X', Z', Y') \in R_G$ ($|Z'| = k - 1 < n - 2$)

Subcase 1: $X' \cup Y' \cup Z' = V$ then either $|X'| \geq 2$ or $|Y'| \geq 2$. If $|Y'| \geq 2$, $Y' = Y'' \cup c$ then

$$R_G(X', Z', Y'' \cup c) \rightarrow R_G(X', Z', c) \wedge R_G(X', Z', Y'): \quad X.2.1$$

$$\begin{array}{lllll} \Rightarrow & R_G(X', Z' \cup Y'', c) & \wedge & R_G(X', Z' \cup c, Y'') & : \text{X.3} \\ \Rightarrow & R(X', Z' \cup Y'', c) & \wedge & R(X', Z' \cup c, Y'') & : \text{Induction} \\ \Rightarrow & R(X', Z', Y'' \cup c) & = & R(X', Z', Y') & : \text{X.4.1} \end{array}$$

If $|X'| \geq 2$, use a similar argument relying on X.2.2 and X.4.2 instead of X.2.1 and X.4.1.

Subcase 2 $X' \cup Y' \cup Z' \overset{\subset}{\neq} V$

Let $c \notin X' \cup Y' \cup Z'$

$$R_G(X', Z', Y') \rightarrow R_G(c, Z', Y') \vee R_G(X', Z', c) : X.5$$

$$\begin{array}{lllll} \Rightarrow & R_G(X', Z' \cup c, Y') & \wedge & [R_G(c, Z' \cup X', Y') \vee R_G(X', Z' \cup Y', c)] & \\ & \text{by strong union} & & & \\ \Rightarrow & R(X', Z' \cup c, Y') & \wedge & [R(c, Z' \cup X', Y') \vee R(X', Z' \cup Y', c)] & \\ & \text{by induction} & & & \\ \Rightarrow & R(X' \cup c, Z', Y') & \vee & R(X', Z', Y' \cup c) & \\ & \text{by X.4.2 and X.4.1} & & & \\ \Rightarrow & R(X', Z', Y') & \vee & R(X', Z', Y') = R(X', Y', Z') & \\ & \text{by X.2.1 andX.2.2} & & & \end{array} \quad (1)$$

4 Proof of D-Mapness (Induction on $|Z|$)

Basis: If $t = (a, V/\{a, b\}, b) \in R$ then $(a \rightarrow b) \notin E_G \Rightarrow t \in G_0$

Step: Due to Lemma 2, enough to prove for t's with singletons on the left and right.

Let $t = (a, Z, b) \in R \Rightarrow t \in R_G$ for all $|Z| = k \leq n - 2$. Let $t' = (a', Z', b') \in R$ and $|Z'| = k - 1 < n - 2$.

$\Rightarrow \exists c, c \notin Z' \cup \{a', b'\}$. By transitivity

$$R(c, Z', b') \vee R(a', Z', c)]$$

$$\Rightarrow R(a', Z' \cup c, b') \wedge [R(c, Z' \cup a', b') \vee R(a', Z' \cup b', c)]$$

by X.3

$$\Rightarrow R_G(a', Z' \cup c, b') \wedge [R_G(c, Z' \cup a', b') \vee R_G(a', Z' \cup b', c)]$$

by induction

$$\Rightarrow R_G(c \cup a', Z', b') \wedge R_G(a', Z', b' \cup c)$$

by X.4.2 and X.4.1

$$\Rightarrow R_G(a', Z', b') \vee R_G(a', Z', b') = R_G(a', Z', b')$$

by X.2.2 and X.2.1 $\square$

References

[PPU] Paz, A., Pearl, J., and Ur, S., "A New Characterization of Graphs Based on Interception Relations," Computer Science Department, Aarhus University, ISSN 0105-8517, DAIMI PB-444, June 1993.