Computer Science Department Technical Report
University of California
Los Angeles, CA 90024-1596

NP-COMPLETE PROBLEMS IN OPTIMAL HORN CLAUSES SATISFIABILITY

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July 1991
CSD-910028

TECHNICAL REPORT
R-158
January 1991

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February 22, 1991

1 Introduction

A *Horn formula* is a propositional formula in a conjunctive normal form $D_1 \wedge \ldots \wedge D_n$, where each D_i is a disjunctions of literals (propositional letters or negations of propositional letters), and each D_i contains at most one positive literal. A *Horn clause* is a disjunction of literals with at most one positive literal.

In this report we will show that the following problems are NP-Complete :

Maximum Horn Set Satisfiability.

Instance: Set U of variables, a *set* S of Horn clauses over U, positive integer $K \leq |S|$.

Question: Is there a truth assignment for U that simultaneously satisfies at least K of the clauses in S ?

Maximum 2-Horn Satisfiability.

Instance: Set U of variables, a *collection* C of Horn clauses over U such that each clause $c \in C$ has $|c| \leq 2$, positive integer $K \leq |C|$.

Question: Is there a truth assignment for U that simultaneously satisfies at least K of the clauses in C ?

Horn Clauses Optimization.

Instance: Set U of variables, a *set* S of Horn clauses over U, such that each clause $c \in S$ has $|c| \leq 2$, a set of positive integers P, a function $f : S \rightarrow P$ and a positive integer K such that $K \leq \sum_{c \in S} f(c)$.

Question: Is there a truth assignment for U that simultaneously satisfies a set $S' \subseteq S$ of clauses such that $\sum_{c \in S'} f(c) \geq K$?

Constrained Horn Clauses Optimization.

Instance: Set U of variables, a *set* S of Horn clauses over U, such that each clause $c \in S$ has $|c| \leq 2$, two clauses c'_1 and c'_2 such that $|c'_j| \leq 2$ for $j \in \{1, 2\}$, a set of positive integers P, a function $f : S \rightarrow P$ and a positive integer K such that $K \leq \sum_{c \in S} f(c)$.

Question: Is there a truth assignment for U that simultaneously satisfies c'_1 and c'_2 and a set $S' \subseteq S$ of clauses such that $\sum_{c \in S'} f(c) \geq K$?

The proofs of the NP-Completeness of *Maximum Horn Set Satisfiability* and *Maximum 2-Horn Satisfiability* are done by reducing 3-Satisfiability to them (see [GJ79] for definition of *3-satisfiability*) and are variations of the proof given by Garey et al. for the NP-Completeness of *Maximum 2-Satisfiability* (see [GJS76]). We reduce *Maximum 2-Horn Satisfiability* to *Horn Clauses Optimization* to show that the latter is NP-Complete, and NP-Completeness of *Constrained Horn Clauses Optimization* is proved by reducing *Horn Clauses Optimization* to it.

We would like to point out that there are *linear* algorithms to test the satisfiability of a set of Horn Clauses, i.e. , to test whether there is a truth assignment to the set of variables that appear in the clauses that simultaneously satisfies *all* the clauses in the set (see [DG84]). In light of this result, it is easy to realize that all the above problems are in NP.

2 Proofs

Theorem 1 Maximum Horn Set Satisfiability *is NP-Complete.*

Proof: We have seen that *Maximum Horn Set Satisfiability* is in NP. We will show that there is a polynomial transformation from 3-Satisfiability to this problem, and hence, that it is NP-Complete.

Suppose we are given a formula α in 3-CNF ,i.e., α is a conjunction of disjunctive clauses, each having at most 3 literals. We label the collection of all those clauses C. Without loss of generality, We assume that each c_i in C is in one of the following forms (where P, Q and R are propositional letters):

form 0 : $P \vee Q \vee R$

form 1 : $\neg P \vee Q \vee R$

form 2 : $\neg P \vee \neg Q \vee R$

form 3 : $\neg P \vee \neg Q \vee \neg R$

We will define R_{c_i} - the set of Horn clauses that 'represents' a clause $c_i \in C$ as follows:

- If c_i is of form 0 or form 1, then

$$R_{c_i} = \left(\bigcup_{j=1}^{10}\{G_{i,j}\}\right)\bigcup \; \{ \; (p \vee \neg G_{i,1}), (Q \vee \neg G_{i,2}), (R \vee \neg G_{i,3}), (D_i \vee \neg G_{i,4}), \\ (\overline{p} \vee \neg Q \vee \neg G_{i,5}), (\overline{p} \vee \neg R \vee \neg G_{i,6}), (\neg R \vee \neg Q \vee \neg G_{i,7}), \\ (p \vee \neg D_i \vee \neg G_{i,8}), (Q \vee \neg D_i \vee \neg G_{i,9}), (R \vee \neg D_i \vee \neg G_{i,10})\}$$

where if c_i is of form 0 then $p = P$ and $\overline{p} = \neg P$ and if c_i is of form 1 then $p = \neg P$ and $\overline{p} = P$. All the variables in $\{D_i\} \bigcup (\bigcup_{j=1}^{10}\{G_{i,j}\})$ are assumed to be new and disjoint.

The reader can verify that if a truth assignment M satisfies c_i then there is a truth assignment M' that satisfies exactly 17 clauses in R_{c_i} such that $M(a) = M'(a)$ for all $a \in \{P, Q, R\}$. Moreover, there is no truth assignment that satisfies more than 17 clauses in R_{c_i} , and if a truth assignment does not satisfy c_i, then it satisfies at most 16 clauses in R_{c_i}.

- If c_i is of form 2 or 3, then

$$R_{c_i} = \{G_i\} \bigcup \{\neg P \vee \neg Q \vee r \vee \neg G_i\}$$

Where $r = R$ if c_i is of form 2 and $r = \neg R$ if c_i is of form 3, and G_i is a new variable.

The reader can verify that if a truth assignment M satisfies c_i then there is a truth assignment M' that satisfies exactly 2 clauses in R_{c_i} such that $M(a) = M'(a)$ for all $a \in \{P, Q, R\}$. Moreover, there is no truth assignment that satisfies more than 2 clauses in R_{c_i} , and if a truth assignment does not satisfy c_i, then it satisfies at most 1 clauses in R_{c_i}.

Note that for each $i \neq j$, $R_{c_i} \bigcap R_{c_j} = \emptyset$.

Let l be the number of clauses in C of form 0 or 1, and let m be the number of clauses in C of form 2 or 3. We claim that α is satisfiable iff there is a truth assignment that satisfies at least $k = 17l + 2m$ clauses in the set $S = \bigcup_{c_i \in C} R_{c_i}$. For if α is satisfiable, then there is a truth assignment that satisfies all the clauses in C, and so there must exist a truth assignment that satisfies at least k clauses from S and if there is a truth assignment that satisfies at least k clauses from S , then this truth assignment must satisfy every clause in C. □

Theorem 2 Maximum 2-Horn Satisfiability *is NP-Complete.*

Proof: We have shown that *Maximum 2-Horn Satisfiability* is in NP. We will show that there is a polynomial transformation from 3-Satisfiability to this problem and hence it is NP-Complete. We will use the notations and definitions of the previous proof without repeating them.

Suppose we are given a formula α in 3-CNF - i.e. - α is a conjunction of disjunctive clauses, each having at most 3 literals. We label the collection of all those clauses C. Without loss of generality, We assume that each c_i is of form 0,1,2 or 3.

We will define R_{c_i} - the collection of Horn clauses that 'represents' a clause $c_i \in C$ as follows:

- If c_i is of form 0 or 1, then

$$\begin{aligned} R_{c_i} &= \{(p),(Q),(R),(D_i), \\ &(\overline{p} \vee \neg Q),(\overline{p} \vee \neg R),(\neg Q \vee \neg R), \\ &(p \vee \neg D_i),(Q \vee \neg D_i),(R \vee \neg D_i)\} \end{aligned}$$

where if c_i is of form 0 then $p = P$ and $\overline{p} = \neg P$ and if c_i is of form 1 then $p = \neg P$ and $\overline{p} = P$. The same collection of clauses appears in Garry et al. proof for the NP-Completeness of *Maximum 2-Satisfiability* (see [GJS76]).

- If c_i is of form 2, then

$$\begin{aligned} R_{c_i} &= \{(R),(G_i),(D_i), \\ &(\neg R \vee P),(\neg R \vee Q),(\neg P \vee \neg Q), \\ &(R \vee \neg D_i),(\neg P \vee \neg D_i),(\neg Q \vee \neg D_i)\} \end{aligned}$$

- If c_i is of form 3, then

$$\begin{aligned} R_{c_i} &= \{(P),(Q),(R),(D_i), \\ &(\neg R \vee \neg P),(\neg R \vee \neg Q),(\neg P \vee \neg Q), \\ &(\neg R \vee \neg D_i),(\neg P \vee \neg D_i),(\neg Q \vee \neg D_i)\} \end{aligned}$$

The reader can verify that no matter what is the form of c_i, if a truth assignment M satisfies c_i then there is a truth assignment M' that satisfies exactly 7 clauses in R_{c_i} such that $M(a) = M'(a)$ for all $a \in \{P, Q, R\}$. Moreover, there is no truth assignment that satisfies more than 7 clauses in R_{c_i} , and if a truth assignment does not satisfy c_i, then it satisfies at most 6 clauses in R_{c_i}.

α is satisfiable iff there is a truth assignment that satisfies at least $k = 7|C|$ clauses in the collection $H = \bigcup_{c_i \in C} R_{c_i}$. Note that for each $c_i \in H$, $| c_i | \leq 2$. □

Theorem 3 Horn Clauses Optimization *is NP-Complete.*

Proof: We have seen that the problem is in NP. We will show that there is a polynomial transformation from *Maximum 2-Horn Satisfiability* to this problem. Suppose we are given a collection C of horn clauses of size ≤ 2, and an integer K, and we want to know if there is a truth assignment that satisfies at least K clauses from this collection . For each $c_i \in C$, let $\#c_i$ denote the number of times c_i appears in C, and let S be the set of clauses that appear in C. Let Max be $\max_{c_i \in C}\{\#c_i\}$, and Let $P = \{1, \ldots, Max\}$. We will define $f(c_i) = \#c_i$.

There is a truth assignment that satisfies at least K clauses from C iff there is a truth assignment that simultaneously satisfies a set $S' \subseteq S$ such that $\sum_{c_i \in C} f(c_i) \geq K$. □

Theorem 4 Constrained Horn Clauses Optimization *is NP-Complete.*

Proof: As mentioned before, this problem is in NP. We will show that if there is a polynomial algorithm A to solve this problem, then there must be a polynomial algorithm A' to solve *Horn Clause Optimization.*

Suppose we are given a *Horn Clauses Optimization* problem . To find if there is a truth assignment that simultaneously satisfies a set $S' \subseteq S$ s.t. $\sum_{c \in S'} f(c) \geq K$, We will first check if there is one clause c in S with $f(c) \geq K$. If we find one . we are done. If there is not such a clause, We will continue by running A $|S| * (|S| - 1)$ times, each time selecting a different set of 2 clauses from S as c'_1 and c'_2 , and testing whether there is an assignment that satisfies c'_1 and c'_2 and a set $S'' \subseteq S - \{c'_1, c'_2\}$ such that $\sum_{c \in S''} f(c) \geq K - (f(c'_1) + f(c'_2))$. □

References

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