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RADIX-4 SQUARE ROOT WITHOUT INITIAL PLA

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Abstract

A radix-4 square root algorithm using redundancy in the partial residuals and the result is presented. Unlike other similar scheme it does not use a table-lookup or PLA for the initial step. The scheme can be integrated with division. It performs on-the-fly conversion and rounding of the result, thus eliminating a carry-propagate step to obtain the final result. The selection function uses 4 bits of the result and 8 bits of the estimate of the partial residual. The paper includes a systematic derivation of the algorithm.

1. Introduction

Several implementations of radix-4 square root have been presented in the literature [VINE65, GOSL87, FAND87, ZURA87]. In all these cases a table-lookup or a special PLA is included for the determination of the first few bits of the result, while another PLA implements the result-digit selection for the remaining radix-4 digits. In this paper we show that this initial PLA is not necessary, resulting in a simpler implementation. As in the other designs, the implementation can be combined with division: the result digit selection function and the recurrence are identical in all steps.

The operand and result are in floating-point representation. To permit the computation of the exponent of the result, the exponent of the operand has to be even. To accomplish this, the mantissa of the operand is multiplied by 1/2 when its exponent is odd. Consequently, the operand mantissa is in the range [1/4,1). The mantissa of the result is then in the range [1/2,1).

To obtain a fast implementation, as done in [GOSL87, FAND87, ZURA87], carry-save addition is used and the result-digit selection depends on low-precision estimates of the residual and of the partial result. This requires that the result digit be from a redundant digit-set. As the other referenced implementations we use the set {-2,-1,0,1,2} to simplify the multiple formation required by the recurrence.

The signed-digit result is converted on-the-fly to conventional representation. Moreover, during this conversion on-the-fly rounding is performed [ERCE88].

2. Recurrence and Square Root Step

The algorithm is based on a continued-sum recurrence. We now develop the digit-recurrence for the algorithm and show the implementation of the corresponding iteration step.

2.1 Recurrence

Each iteration of the recurrence produces one digit of the result, most-significant digit first. Let us call $S[j]$ the value of the result after j iterations, that is

$$S[j] = \sum_{i=1}^{j} s_i 4^{-i} \tag{1}$$

The final result is then

$$s = S[n] = \sum_{i=1}^{n} s_i 4^{-i}$$

We define an error function ε so that its value after j steps is

$$\varepsilon[j] = x^{1/2} - S[j] \tag{2}$$

where $1/4 \le x < 1$ is the operand.

To have a correct result this error has to be bounded. To use a redundant representation of the result, we allow a positive or negative error such that

$$-4^{-j} < \varepsilon[j] < 4^{-j} \tag{3}$$

If a positive final remainder is required, a restoration step is included.

We now transform (2) to eliminate the square root operation (add $S[j]$ and obtain the square). We get

$$4^{-2j} - 2\times 4^{-j} S[j] + S[j]^2 < x < 4^{-2j} + 2\times 4^{-j} S[j] + S[j]^2$$

Subtracting $S[j]^2$ we obtain

$$4^{-2j} - 2\times 4^{-j} S[j] < x - S[j]^2 < 4^{-2j} + 2\times 4^{-j} S[j] \tag{4}$$

That is, we have to compute $S[j]$ such that $x - S[j]^2$ is bounded according to (4). We now define a residual (or partial remainder) w so that

$$w[j] = 4^j (x - S[j]^2) \tag{5}$$

From (4) the bound on the residual is

$$-2S[j] + 4^{-j} < w[j] < 2S[j] + 4^{-j} \tag{6}$$

and its initial condition

$$w[0] = x \tag{7}$$

In terms of this residual we obtain the recurrence

$$w[j+1] = 4w[j] - 2S[j]s_{j+1} - s_{j+1}^2 \times 4^{-(j+1)} \tag{8}$$

Expression (8) is the basic recurrence on which the square root algorithm is based.

2.2 Implementation of Square root step

The square root algorithm consists in performing m iterations of the recurrence (8). Moreover, each iteration consists of four subcomputations (Figure 1a):

1) One digit arithmetic left-shift of $w[j]$ to produce $4 \cdot w[j]$.

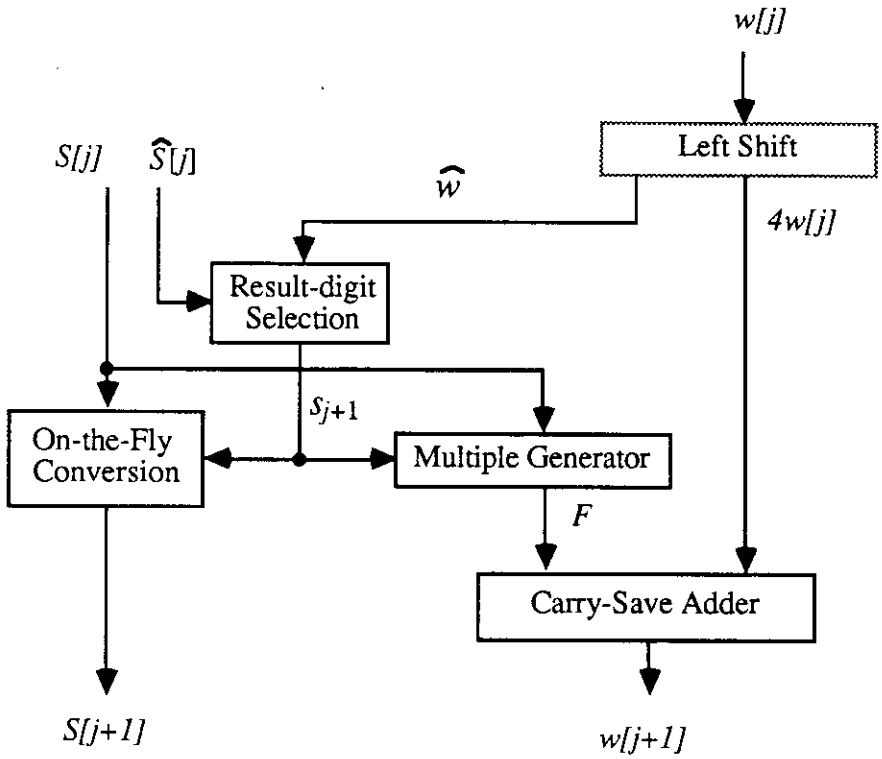
2) Determination of the result digit s_{j+1}. To do this a *result-digit selection function* is used. The value of the digit s_{j+1} is selected so that the application of the recurrence produces a $w[j+1]$ that satisfies the bound (6). The function has as arguments $\hat{w}[j]$ (an estimate of $4w[j]$) and $\hat{S}[j]$ (an estimate of $S[j]$) and produces s_{j+1}. That is,

$$s_{j+1} = f(\hat{w}[j], \hat{S}[j]) \tag{9}$$

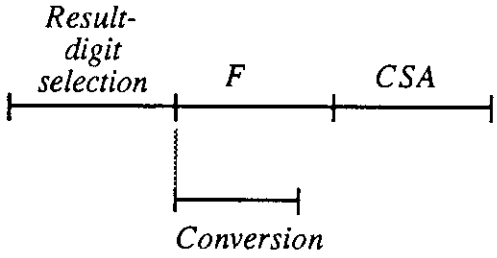
3) Formation of $F = -2S[j]s_{j+1} - s_{j+1}^2 4^{-(j+1)}$ and $S[j+1] = S[j] + s_{j+1}4^{-(j+1)}$.

4) Subtraction of F from $4w[j]$ to produce $w[j+1]$.

The four subcomputations are executed in sequence as indicated in the timing diagram of Figure 1b. Note that no time has been allocated for the arithmetic shift since it is performed just by suitable wiring. Moreover, the relative magnitudes of the delay of each of the components depend on the specific implementation.



(a)



(b)

Figure 1. (a) Square root step, (b) Timing

To have a fast recurrence step we use a carry-save adder and a result-digit selection function that depends on low-precision estimates of the residual and of the partial result. To achieve this, it is necessary to have a redundant representation of the result-digit. In particular, we use the symmetric signed-digit set

$$s_j \in \{-2,-1,0,1,2\} \tag{10}$$

because it allows a simpler implementation of the adder input F. Moreover, the signed result-digit makes it necessary to use an on-the-fly conversion to produce $S[j]$ in a conventional form for the formation of F.

2.3 Implementation of Square Root Algorithm

As indicated, the square root algorithm consists of m iterations of the recurrence. The implementation of this algorithm can be *totally sequential*, where the hardware of the step is reused for all the iterations and the residual is updated in a register (Figure 2a); *totally combinational*, where the hardware for the step is replicated (Figure 2b); or a combination of both, where the step hardware is replicated k times and this superstep is reused m/k times (Figure 2c). Specially in the combinational implementations, pipelining can be used so that several operations can use the hardware at the same time, with the corresponding increase in throughput. The selection of one of these alternatives is influenced by cost, speed, and throughput considerations.

3. Result-digit Selection Function

We now present the design of the result-digit selection function. This function determines the value of the result digit s_{j+1} as a function of the residual $w[j]$ and the partial result $S[j]$.

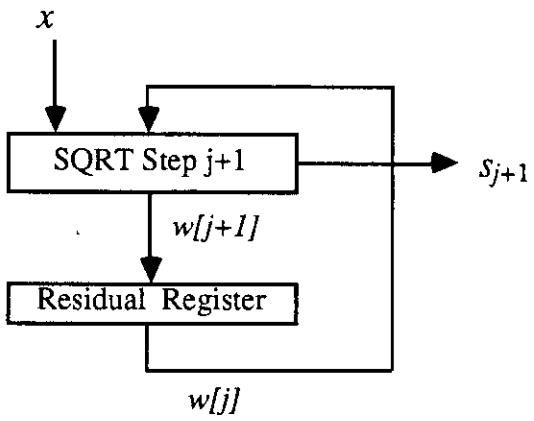
There are two fundamental conditions that must be satisfied by a selection function: *containment* and *continuity*. These conditions determine a *selection interval* for each value of s_{j+1}, from which alternative result-digit selections can be defined.

3.1 Containment Condition and Selection Intervals

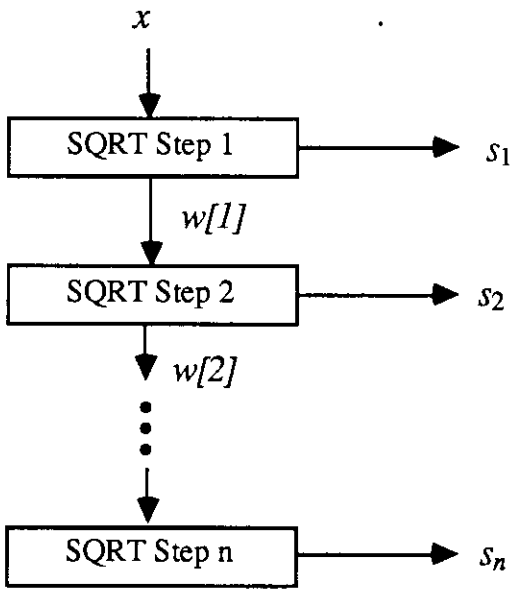
One basic requirement for the result-digit selection is that the selection has to be made in a way that produces a next residual that is bounded. This leads to the containment condition, which we now develop.

Let the bounds of the residual $w[j]$ be called $\underline{B}$ and $\bar{B}$, that is,

$$\underline{B}[j] \le w[j] \le \bar{B}[j] \tag{11}$$

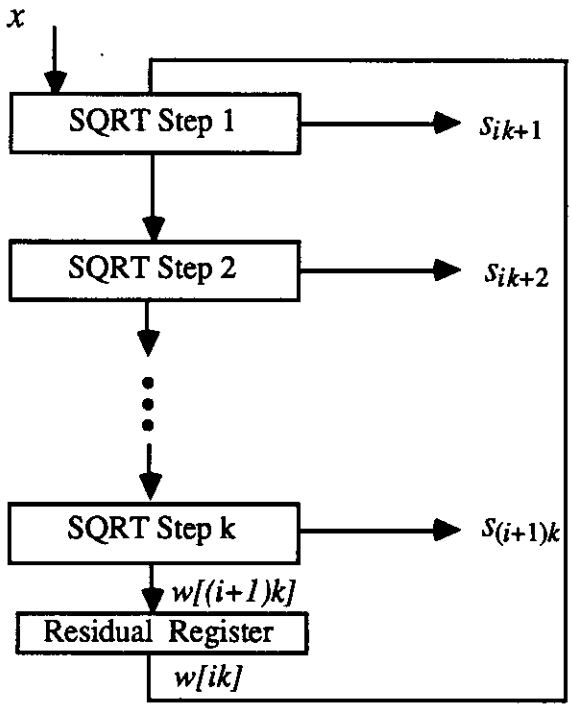


(a)



(b)

Figure 2 Square root implementation: (a) Totally sequential , (b) Totally combinational



(c)

Figure 2 cont. (c) Combined implementation

Note that these bounds depend on j, unlike in division, where they are constants.

Define the *selection interval* of $4 \cdot w[j]$ for $s_j = k$ to be $[L_k, U_k]$. That is, L_k (U_k) is the smallest (largest) value of $4 \cdot w[j]$ for which it is *possible* to choose $s_{j+1} = k$ and keep $w[j+1]$ bounded. Therefore,

$$L_k[j] \le 4w[j] \le U_k[j] \;\rightarrow\; \underline{B}[j+1] \le 4w[j] - 2S[j]k - k^2 4^{-(j+1)} \le \overline{B}[j+1] \tag{12}$$

Consequently,

$$U_k[j] = \overline{B}[j+1] + 2S[j]k + k^2 4^{-(j+1)} \qquad L_k[j] = \underline{B}[j+1] + 2S[j]k + k^2 4^{-(j+1)} \tag{13}$$

We now can determine $\overline{B}[j]$ and $\underline{B}[j]$ because they are the upper bound of the interval for $k=2$ and the lower bound for $k=-2$, respectively. That is,

$$U_2[j] = 4\overline{B}[j] \qquad L_{-2}[j] = 4\underline{B}[j]$$

Introducing these values in (13) we get

$$\begin{aligned} &\overline{B}[j+1] + 2S[j]\times 2 + 2^2 4^{-(j+1)} = 4\overline{B}[j] \\ &\underline{B}[j+1] - 2S[j]\times 2 + 2^2 4^{-(j+1)} = 4\underline{B}[j] \end{aligned} \tag{14}$$

This results in

$$\begin{aligned} \overline{B}[j] &= \frac{4}{3}S[j] + \frac{4}{9}4^{-j} \\ \underline{B}[j] &= -\frac{4}{3}S[j] + \frac{4}{9}4^{-j} \end{aligned} \tag{15}$$

To show that (15) is correct it is sufficient to replace in (14). Note that, in contrast to division, the bounds vary with j. These bounds satisfy the bound on the residual of (6).

The *containment condition* is obtained from (13) and (15). It states that the selection interval for $s_{j+1} = k$ is given by the expressions

$$U_k[j] = \frac{4}{3}S[j+1] + \frac{4}{9}4^{-(j+1)} + 2S[j]k + k^2 4^{-(j+1)}$$

Since $S[j+1] = S[j] + k \times 4^{-(j+1)}$ we get

$$U_k[j] = 2S[j](k + \frac{2}{3}) + (k + \frac{2}{3})^2 4^{-(j+1)} \tag{16a}$$

Similarly,

$$L_k[j] = 2S[j](k - \frac{2}{3}) + (k - \frac{2}{3})^2 4^{-(j+1)} \tag{16b}$$

The square-root recurrence, the residual bounds, and the selection-interval bounds can be represented in **Robertson's diagram** (Figure 3). This diagram has as axes the shifted residual $4w[j]$ and the next residual $w[j+1]$. It represents the recurrence by the lines with parameter $s_{j+1} = k$ for $k = -2,...,2$ and the residual bounds by the rectangle $w[j+1] = \bar{B}[j+1]$, $w(j+1) = \underline{B}[j+1]$, $4w[j] = 4\bar{B}[j]$, and $4w[j] = 4\underline{B}[j]$. The selection interval for $s_{j+1} = k$ is obtained from the projection of the corresponding line on the $4w[j]$ axis.

Another diagram that contains information more useful in the design of the result-digit selection function is the residual vs. partial result plot, called the *R-PR plot* (Figure 4). It is similar to the *P-D plot*, used in division: it has as axes the partial result $S[j]$ and the shifted residual $4w[j]$. The bounds of the selection intervals U_k and L_k are ploted as lines.

3.2 Continuity Condition and Overlap Between Selection Intervals

A second requirement for the selection intervals is the *continuity condition*. It states that for any value of $4w[j]$ between $4\underline{B}[j]$ and $4\bar{B}[j]$ it must be possible to select *some* value for the result digit. This can be expressed as

$$U_{k-1} \geq L_k - 4^{-m} \tag{17}$$

Moreover, to use estimates of $4w[j]$ and of $S[j]$ for the result-digit selection, it is necessary to have an overlap between the adjacent selection intervals. For the square-root operation with digit-set {-2,-1,0,1,2} the overlap is

$$U_{k-1} - L_k = \frac{1}{3}(2S[j] + (2k-1)4^{-(j+1)}) \tag{18}$$

Note that the overlap depends on $S[j]$, on k, and on j. We will analyze the different cases later and show that there is sufficient overlap to use estimates for the result-digit selection.

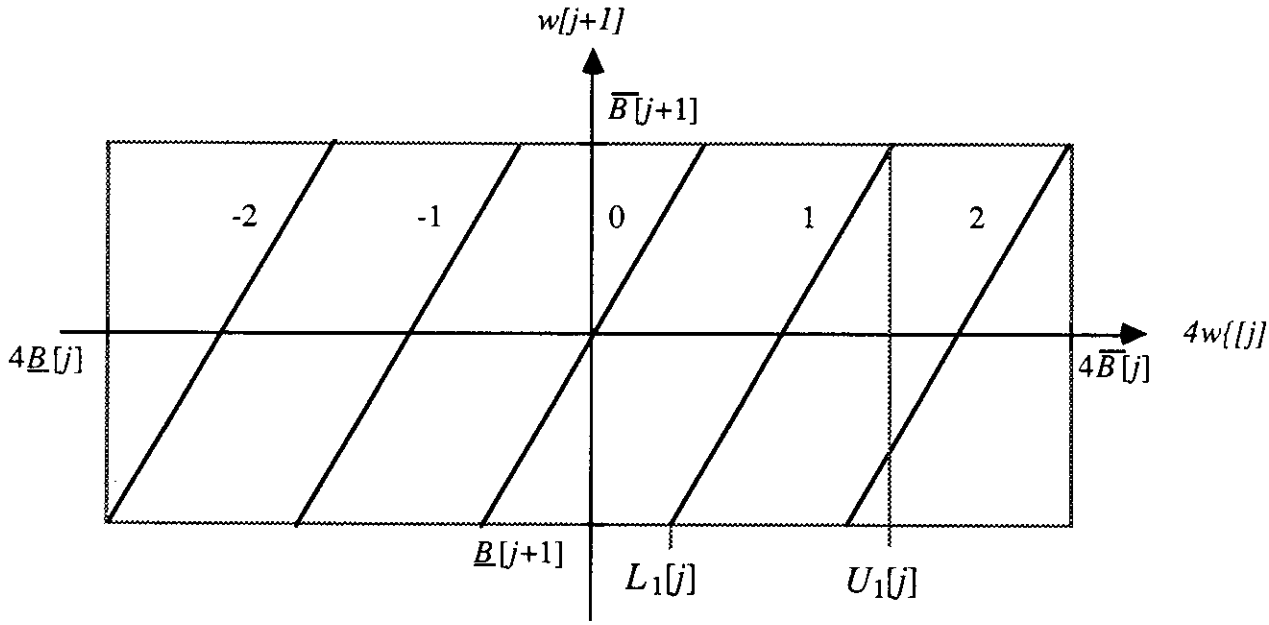


Figure 3. Robertson's diagram

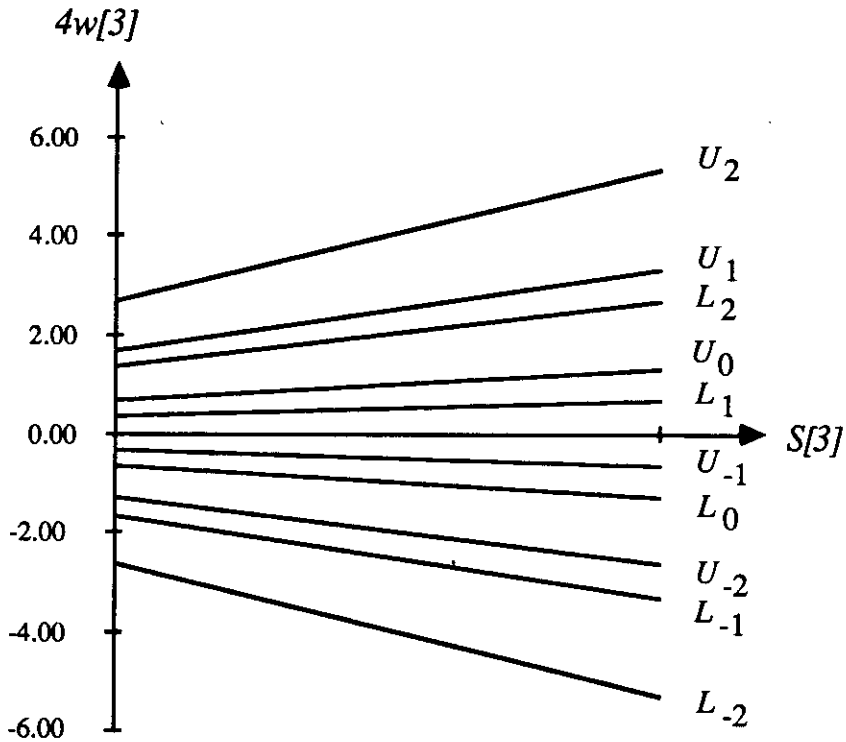


Figure 4. PD Diagram (j=3)

3.3 Result-digit selection for residual in carry-save form

We now determine the result-digit selection function using an estimate of the shifted residual obtained by truncating the carry-save form.

The truncation of the shifted residual in carry-save form to t fractional bits produces an estimate $\hat{w}$ with error satisfying

$$0 \le 4w[j] - \hat{w} < 2 \times 2^{-t} \tag{19}$$

as shown in Figure 5.

Consequently, to have a correct result the basic requirement is that if we choose $s_{j+1}=k$ for an estimate $\hat{w}$, then this selection has to be acceptable for the interval

$$4w[j] \in [\hat{w}, \hat{w}+2^{-(t-1)}] \tag{20}$$

The result-digit selection function we develop is of the "staircase" type as illustrated in Figure 6a. Such a function is defined by selection constants $m_i(k)$ which are used for partial result interval $S[j] \in [S_i, S_{i+1})$, where $S_i = (2^{-1} + i \times 2^{-\delta})$. That is, for that interval we choose

$$s_{j+1} = k \quad \text{if} \quad m_i(k) \le \hat{w} < m_{i+1}(k)$$

The set $\{m_i(k) \mid 0 \le i \le (2^{\delta-1}-1)$ and $-2 \le k \le 2\}$ defines the result-digit selection function.

If the selection constants have a precision of t fractional bits, that is

$$m_i(k) = A_i(k)2^{-t}$$

where $A_i(k)$ is an integer, we get from the containment and continuity conditions, and (20)

$$A_i(k)2^{-t} \ge max\,(L_k(S_i), L_k(S_{i+1})) \tag{21a}$$

$$(A_i(k) - 1)2^{-t} + 2^{-(t-1)} \le min\,(U_{k-1}(S_i), U_{k-1}(S_{i+1})) \tag{21b}$$

The second expression has to hold because for $(A_i(k)-1)$ we want to choose $s_{j+1} = k-1$. These expressions are illustrated in Figure 6b.

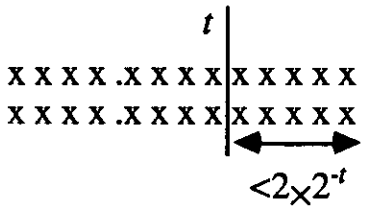
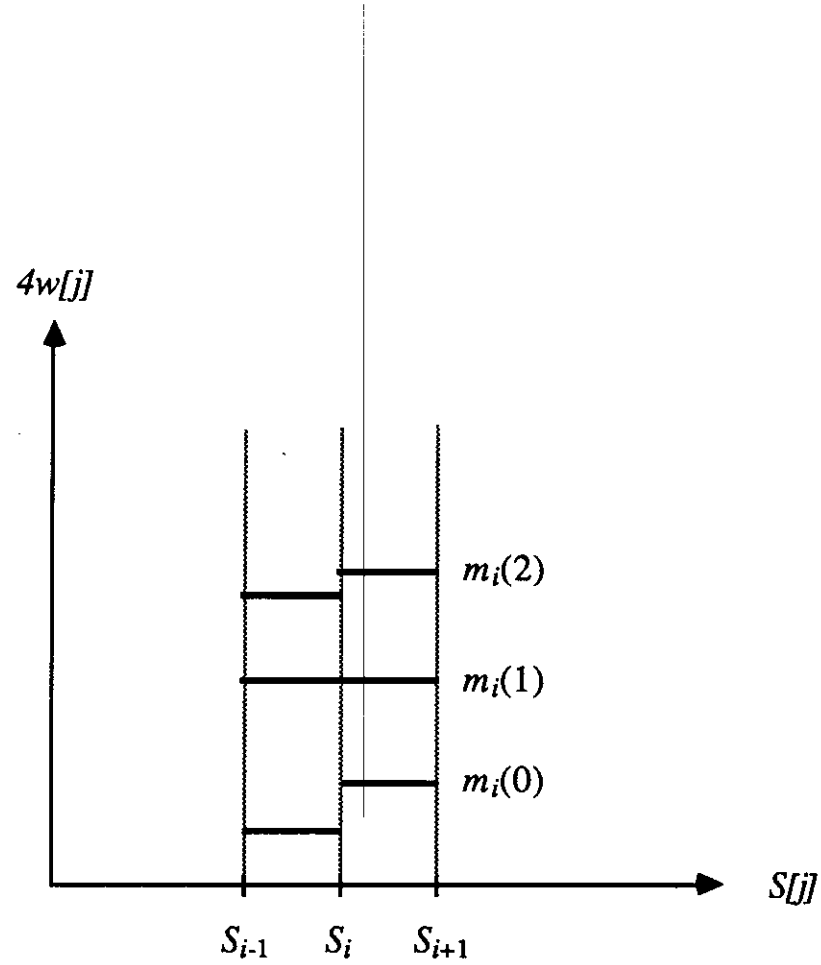


Figure 5. Truncation error



(a)

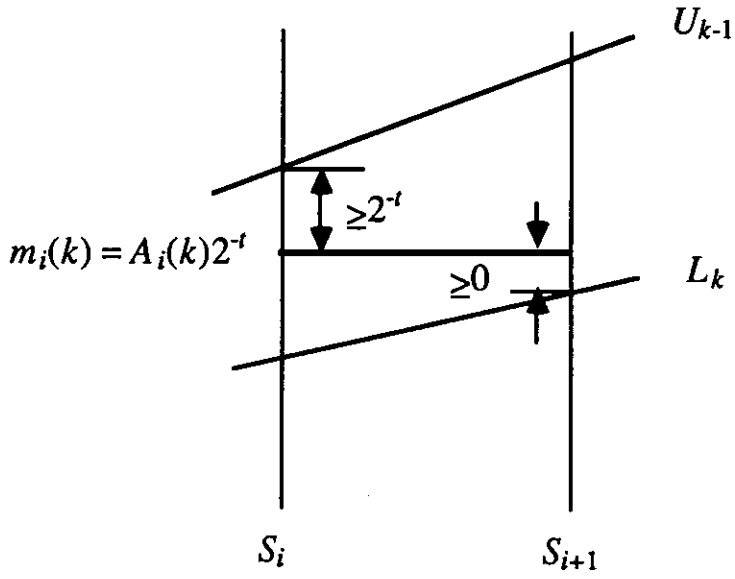


Figure 6. (a) Staircase selection - a fragment,
(b) Conditions on selection constants

Consequently, the main relation used for obtaining the corresponding (staircase) result-digit selection is

$$A_i(k)2^{-t} \geq max\,(L_k(S_i), L_k(S_{i+1})) \qquad (22)$$

$$A_i(k)2^{-t} \leq min\,(\hat{U}_{k-1}(S_i), \hat{U}_{k-1}(S_{i+1}))$$

where $\hat{U} = U - 2^{-t}$.

The values of δ and t are obtained by trial and error. To reduce the number of trials, lower bounds are obtained from the need for a sufficient overlap, as described in [ERCE89].

For the radix-4 case with digit-set {-2,...,2} we have from (16)

$$U_2[j] = 2S[j]\times(8/3) + (8/3)^2 4^{-(j+1)} \qquad L_2[j] = 2S[j]\times(4/3) + (4/3)^2 4^{-(j+1)}$$

$$U_1[j] = 2S[j]\times(5/3) + (5/3)^2 4^{-(j+1)} \qquad L_1[j] = 2S[j]\times(1/3) + (1/3)^2 4^{-(j+1)}$$

$$U_0[j] = 2S[j]\times(2/3) + (2/3)^2 4^{-(j+1)} \qquad L_0[j] = -2S[j]\times(2/3) + (2/3)^2 4^{-(j+1)}$$

$$U_{-1}[j] = -2S[j]\times(1/3) + (1/3)^2 4^{-(j+1)} \qquad L_{-1}[j] = -2S[j]\times(5/3) + (5/3)^2 4^{-(j+1)}$$

$$U_{-2}[j] = -2S[j]\times(4/3) + (4/3)^2 4^{-(j+1)} \qquad L_{-2}[j] = -2S[j]\times(8/3) + (8/3)^2 4^{-(j+1)}$$

Since these limits depend on j, the result-digit selection might vary with the value of j. We now consider the different cases.

Since the maximum digit value is 2, it is necessary to have

$$s_0 = 1$$

to be able to represent values of $s \geq 2/3$. Consequently, $S[0] = 1$, which leads to $s_1 = \{0,-1,-2\}$ (to have $s < 1$).

Therefore, for j=0 we obtain,

$$U_{-1}[0] = -2\times1\times(1/3) + (1/9)(1/4) = -(23/36)$$

$$L_0[0] = -2\times1\times(2/3) + (4/9)(1/4) = -(44/36)$$

This results in a possible $m(0) = -1$.

Similarly,

$$U_{-2}[0] = -2\times1\times(4/3) + (16/9)(1/4) = -(80)/(36)$$

$$L_{-1}[0] = -2\times1\times(5/3) + (25/9)(1/4) = -(95)/(36)$$

which results in a possible $m(-1) = -5/2$. Since $-5/2 = -90/36$ we get a minimum value of $2^{-t} = (90–80)/36 > 1/4$.

For $j = 1$ we can have $s_2 = \{-2,-1,0,1,2\}$. Moreover, since $s_1 = \{-2,-1,0\}$ and $S[0] = 1$ the possible values of $S[1]$ are 1/2, 3/4, 1. A lower bound of t=3 is obtained [ERCE89]. For this value, Table 1 shows the corresponding values of L_k and $\hat{U}_k$ and a result-digit selection function that satisfies (22).

Table 1. Result-digit selection for j=1

	$S[1] = 1/2$	$S[1] = 3/4$	$S[1] = 1$
$L_2, \hat{U}_1$	208/144, 247/144	304/144, 367/144	400/144, 487/144
$m(2)$	3/2	9/4	3
$L_1, \hat{U}_0$	49/144, 82/144	73/144, 130/144	97/144, 178/144
$m(1)$	1/2	3/4	1
$L_0, \hat{U}_{-1}$	-	-140/144, -89/144	-188/144, -113/144
$m(0)$	-	-3/4	-1
$L_{-1}, \hat{U}_{-2}$	-	-335/144, -290/144	-455/144, -386/144
$m(-1)$	-	-9/4	-3

Note that it is not possible to select $s_2 < 0$ when $S[1]$=1/2, because this would make $S[2] < 1/2$.

For j=2 we obtain lower bounds of δ=4 and t=3 (see [ERCE89]). We now develop the result-digit selection using these values. Since we use $\delta = 4$ and the granularity of $S[2]$ is 1/16, we use exact values instead of intervals. Table 2 shows the corresponding values of L_k and $\hat{U}_k$ and a possible result-digit selection. The term $\hat{U}$ is

$$\hat{U}_k = U_k - \frac{1}{8}$$

Table 2. Result-digit selection for j=2

S_i	8/16	9/16	10/16	11/16	12/16	13/16	14/16	15/16	16/16
$L_2, \hat{U}_1 (\times\frac{1}{576})$	784, 913	880, 1033	976, 1143	1072, 1253	1168, 1363	1264, 1473	1360, 1583	1456, 1693	1552, 1803
$m(2)$	3/2	7/4	7/4	2	9/4	5/2	5/2	11/4	3
$L_1, \hat{U}_0 (\times\frac{1}{576})$	193, 316	217, 364	241, 412	265, 460	289, 508	333 556	357, 604	381, 652	405, 700
$m(1)$	1/2	1/2	1/2	1/2	3/4	3/4	1	1	1
$L_0, \hat{U}_{-1} (\times\frac{1}{576})$	-380, -263	-428, -287	-476, -311	-524, -335	-572, -359	-620, -383	-668, -407	-716, -431	-754, -455
$m(0)$	-	-1/2	-3/4	-3/4	-3/4	-1	-1	-1	-1
$L_{-1}, \hat{U}_{-2} (\times\frac{1}{576})$	-935, -824	-1045, -920	-1155,-1016	-1265, -1112	-1375, -1208	-1485, -1304	-1595, -1400	-1705, -1496	-1815, -1592
$m(-1)$	-	-7/4	-2	-2	-9/4	-5/2	-5/2	-11/4	-3

For $j \geq 3$ we use δ=4 and t=4 [ERCE89]. Moreover, to make the function independent of j (only dependent on $S[j]$) we use

$$L_k[j] < 2S[j](k-2/3) + (k-2/3)^2 4^{-4} = L^*_k + \frac{(k-2/3)^2}{256}$$

$$U_k[j] > 2S[j](k+2/3)$$

and

k	2	1	0	-1	-2
$\frac{(k-2/3)^2}{256}$	1/144	1/2304	1/576	1/90	1/36

We get Table 3. For simplicity, in the table we include L^*_k instead of L_k; however, for the result-digit selection it is necessary to take into account the additional term. Because this term is relatively small (with respect to 2^{-t}), the only limitation introduced is that $m_i(k)$ cannot be equal to L^*_k.

Table 3. Result-digit Selection for $j \geq 3$

$[S_i, S_{i+1})$	8/16 9/16	9/16 10/16	10/16 11/16	11/16 12/16	12/16 13/16	13/16 14/16	14/16 15/16	15/16 16/16*
$L^*_2(S_{i+1}), \hat{U}_1(S_i)$	3/2, 77/48	5/3, 29/16	11/6, 97/48	2, 107/48	13/6, 39/16	7/3, 127/48	15/6, 137/48	8/3, 49/16
$m_i(2)$	25/16	7/4	15/8	17/8	9/4	5/2	11/4	3
$L^*_1(S_{i+1}), \hat{U}_0(S_i)$	3/8, 29/48	5/12, 11/16	11/24, 37/48	1/2, 41/48	13/24, 15/16	7/12, 49/48	5/8, 53/48	2/3, 19/16
$m_i(1)$	1/2	1/2	1/2	3/4	3/4	3/4	1	1
$L^*_0(S_i), \hat{U}_{-1}(S_{i+1})$	-2/3, -7/16	-3/4, -23/48	-5/6, -25/48	-11/12, -9/16	-1, -29/48	-13/12, -31/48	-7/6, -11/16	-5/4, -35/48
$m_i(0)$	-1/2	-5/8	-3/4	-3/4	-3/4	-1	-1	-1
$L^*_{-1}(S_i), \hat{U}_{-2}(S_{i+1})$	-5/3, -25/16	-15/8, -83/48	-25/12, -91/48	-55/24, -33/16	-15/6, -107/48	-65/24, -115/48	-35/12, -41/16	-75/24, -133/48
$m_i(-1)$	-13/8	-29/16	-2	-9/4	-9/4	-5/2	-11/4	-3

* includes 16/16

Since we want a single result-digit selection (independent of j), we now need to match the result-digit selections for $j=0$, $j=1$, $j=2$, and $j \geq 3$. We take as a basis the selection for $j \geq 3$ and compare the corresponding entries with those for $j=2$, $j=1$, and $j=0$. When the entries are different we adjust them to satisfy all cases. We get Table 4. The only case we cannot match is the value $m(-1)=-5/2$ for $j=0$ (for $S[0] = 1$), since for the other values of j it is -3. A simple solution is to apply $S[0]=13/16$ instead of 1 for the result-digit selection; this requires 3 AND and 3 OR gates, as shown in Figure 7. Note that the most significant bit of $\hat{S}$ represents the cases $(A_0, A_1) = (1,0)$ and $(A_0, A_1) = (0,1)$. Moreover, to eliminate S_1, we change 1000 to 0111.

Table 4. Single Result-digit Selection (for all values of j)

$[S_i, S_{i+1})$	8/16 9/16	9/16 10/16	10/16 11/16	11/16 12/16	12/16 13/16	13/16 14/16	14/16 15/16	15/16 16/16*
$m_i(2)$	25/16	7/4	15/8	17/8	9/4	5/2	21/8	23/8
$m_i(1)$	1/2	1/2	1/2	3/4	3/4	3/4	1	1
$m_i(0)$	-1/2	-5/8	-3/4	-3/4	-3/4	-1	-1	-1
$m_i(-1)$	-13/8	-29/16	-2	-17/8	-9/4	-5/2	-11/4	-23/8

* includes 16/16

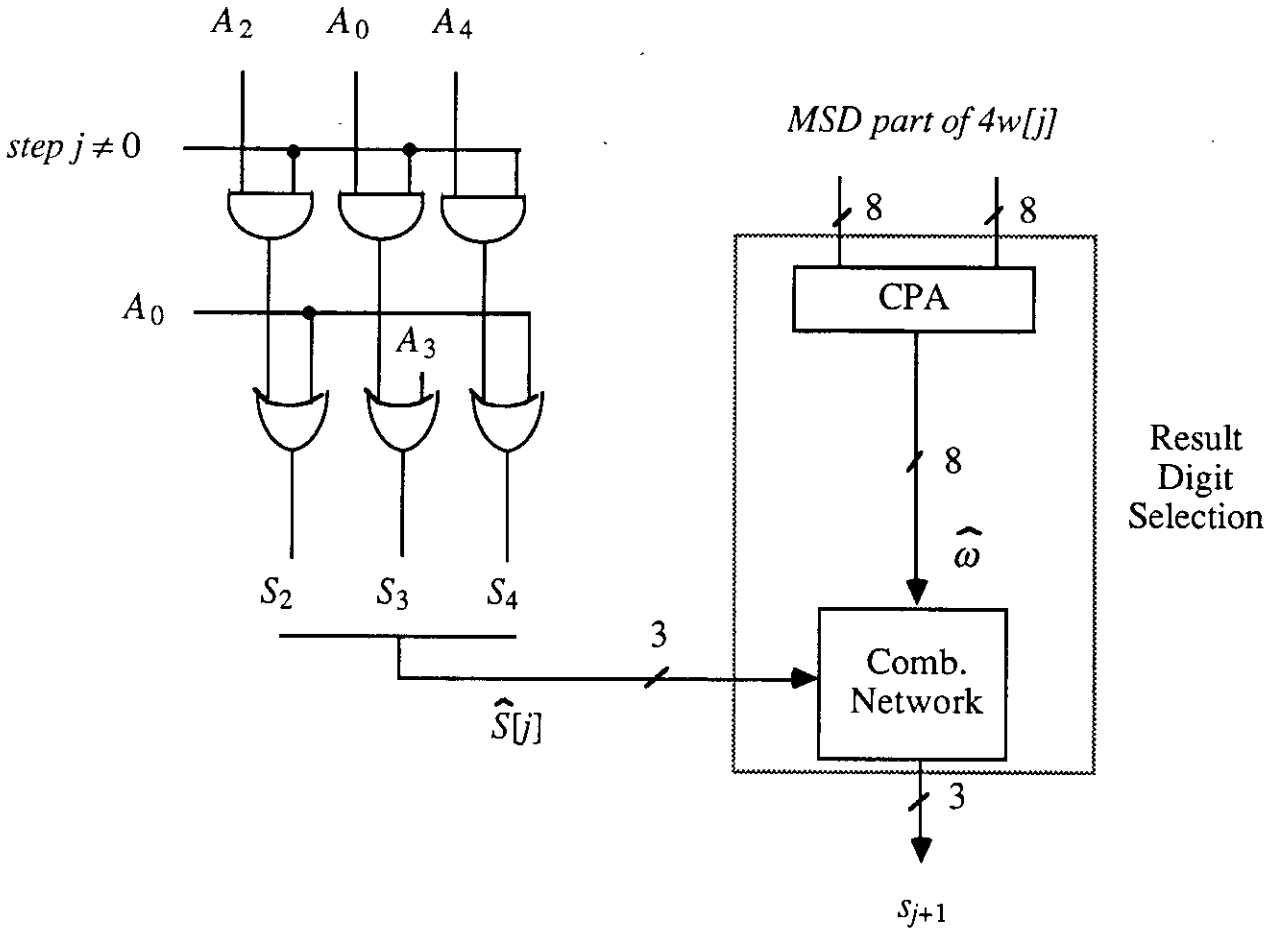


Figure 7. Result-digit selection implementation

4. Generation of adder input F

The third adder input has value

$$F[j] = -2S[j]s_{j+1} - s_{j+1}^2 4^{-(j+1)}$$

To obtain $F[j]$ it is necessary to convert $S[j]$ to conventional radix-2 representation (since s_i is signed-digit). This conversion is done on-the-fly using a variation of the scheme presented in [ERCE87]. It requires that two conditional forms $A[j]$ and $B[j]$ are kept, such that

$$A[j] = S[j]$$

$$B[j] = S[j] - 4^{-j}$$

These forms are updated with each result-digit as follows:

$$A[j+1] = \begin{cases} A[j] + s_{j+1}4^{-(j+1)} & \textbf{if } s_{j+1} \geq 0 \\ B[j] + (4 - s_{j+1})4^{-(j+1)} & \textbf{otherwise} \end{cases}$$

$$B[j+1] = \begin{cases} A[j] + (s_{j+1} - 1)4^{-(j+1)} & \textbf{if } s_{j+1} > 0 \\ B[j] + (3 - s_{j+1})4^{-(j+1)} & \textbf{otherwise} \end{cases}$$

The implementation of this conversion requires two registers for A and B, appending of one digit, and loading. For controlling this appending and loading, a shift register K is used, containing a moving 1. This implementation is shown in Figure 8.

In terms of these forms, the value of F and the corresponding bit-strings are

	$F[j]$	
s_{j+1}	Value	Bit-string
1	$-2A[j] - 4^{-(j+1)}$	$\overline{a} \cdots \overline{aa}111$
2	$-4A[j] - 4\times4^{-(j+1)}$	$\overline{a} \cdots \overline{a}1100$
-1	$2B[j] + 7\times4^{-(j+1)}$	$b \cdots bb111$
-2	$4B[j] + 12\times4^{-(j+1)}$	$b \cdots b1100$

where $a \cdots aa$ and $b \cdots bb$ are the bit-strings representing $A[j]$ and $B[j]$, respectively.

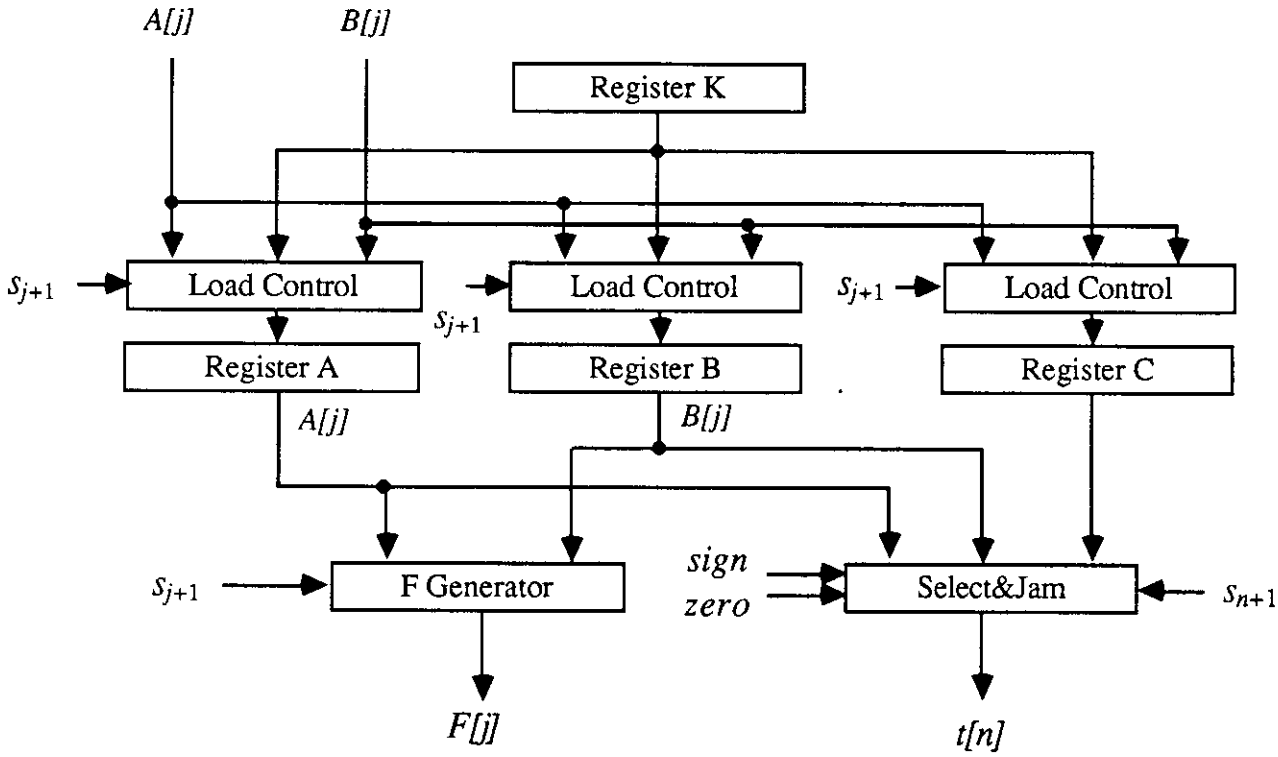


Figure 8. Network for generating F and rounding

Figure 8 shows a block diagram for the generation of F.

5. On-the-fly conversion and rounding

The result digit obtained from the result-digit selection logic is in signed-digit form. As mentioned in the previous section, the partial result is converted on-the-fly to conventional form to use it in the formation of F. Consequently, the final result in conventional form is obtained from register A.

In addition, rounding of the result might be required. The most used type of rounding, rounding-to-nearest, is usually done as follows [FAND87]. First, $n+1$ digits of the result are computed for an n-digit rounded result. Then, a restoration step is performed to obtain a positive residual; to achieve this, the sign of the last residual is determined and the result decremented by one in the least significant position if the sign is negative. Since the representation of the partial residual is redundant (carry-save), the sign has to be obtained from this redundant representation; the process is similar in delay, but simpler in amount of hardware, to a carry-propagate addition that converts the residual to conventional representation. The sign of the residual is then used to decrement the $n+1$-digit result; this can be done by a subtraction or by using the decremented form available from the on-the-fly conversion. Finally, the (unrounded) result is rounded by, possibly, incrementing it by 1. This incrementation requires a carry-propagate addition. The process is costly both in hardware and in time.

To simplify the hardware required for rounding and increase its speed, in [ERCE88] we describe three on-the-fly rounding methods that are combined with the conversion. They are as follows:

1) Rounding to nearest. In this case the first steps of computing an additional digit and finding the sign of the residual are also required. However, neither the restoration step nor the actual rounding require a carry-propagate addition because they can be performed on-the-fly if a third form is computed on-the-fly during the conversion. The method can be summarized as follows. The rounded result with n digits called $t[n]$ is

$$t[n] = \begin{cases} C[n] & \textbf{if} \quad (s_{n+1} - sign) = 2 \\ A[n] & \textbf{if} \quad -1 \le (s_{n+1} - sign) \le 1 \\ B[n] & \textbf{if} \quad (s_{n+1} - sign) \le -2 \end{cases} \tag{4}$$

where $A[n]$ is the converted result with n digits, $B[n] = A[n] - 2^{-n}$, $C[n] = A[n] + 2^{-n}$, and s_{n+1} is the $(n+1)$-nd signed-digit of the result. Moreover, to have unbiased rounding to nearest it is necessary to set to zero the least significant bit of the result when $|s_{n+1}| = 2$ and the last residual (remainder) is zero.

The forms A and B are produced for the conversion as discussed in the previous section. To be able to do the rounding we need to produce also the form C. It is updated as follows:

$$C[j+1] = \begin{cases} A[j] + (s_{j+1} + 1)4^{-(j+1)} & \textbf{if } s_{j+1} \geq -1 \\ B[j] + 3\times 4^{-(j+1)} & \textbf{if } s_{j+1} = -2 \end{cases}$$

This updating is also done by appending and loading, as shown in Figure 8.

In addition, it is necessary to have a network to detect the sign of the remainder from its redundant representation. The most straightforward implementation uses a carry-propagate adder to convert to nonredundant representation; this is especially attractive if a carry-propagate adder exists in the arithmetic unit anyhow for other purposes. In some cases, it might be better to have a special network for this sign detection, either because a carry-propagate adder is not part of the unit or because connection to this adder is slow or complicates the bussing structure. The main component of this sign detection network is the generating of the carry into the last bit; this network follows the standard carry-skip or carry-lookahead techniques.

Finally, the detection of zero remainder is needed for the unbiased rounding to nearest. This detection can follow a conversion of the final residual to irredundant, if such a conversion is used for sign detection, or another special circuit can be used [CORT88].

2) Rounding without sign detection. As a second method for rounding we consider the case in which the sign of the remainder is not detected. This results in a simpler and faster implementation, but with a somewhat larger error. As described in [ERCE88], the rounding rule to produce the minimum error possible (for the digit set $\{-2,...,2\}$) is

$$t[n] = A[n]$$

Consequently, in this case $(n+1)$-th bit of the result does not have to be computed, neither is there need for sign detection, detection of zero, nor computation of C. The rounding is unbiased but with an error bounded by $\pm(2/3)4^{-n}$, which is larger than the error of rounding-to-nearest ($(1/2)4^{-n}$).

3) Rounding with estimate of sign of remainder. As a compromise between the previous two methods it is possible to use an estimate of the sign of the remainder and then use the rounding rules of method 1 above. The estimate is computed using a few most-significant bits of the redundant remainder. More specifically, if k bits of the remainder are used the error is $4^{-n}(2^{-1} + 2^{-k})$. The value of k would be selected to achieve both an acceptable error and a fast and simple implementation. As an example, for $k = 8$, the error is $4^{-n}(2^{-1} + 2^{-8})$, which is less than 1% larger than the error of rounding to nearest. Moreover, the rounding could be performed in one step time instead of in about four, which would be required for full rounding to nearest.

6. Overall implementation and timing

The overall implementation at the block-diagram level is shown in Figure 9. The cycle time is

$$T_{cycle} = t_{result\_digit\_select} \quad \{8\text{–bitCPA} + 12\text{–input network}\}$$
$$+ t_{F-generate} \quad \{4\text{–to–}1 \text{ multiplexer}\}$$
$$+ t_{CSA} \quad \{3\text{–to–}2 \text{ carry–save adder}\}$$
$$+ t_{load} \quad \{\text{register loading}\}$$

This is comparable to the cycle time of a radix-4 division with carry-save adder.

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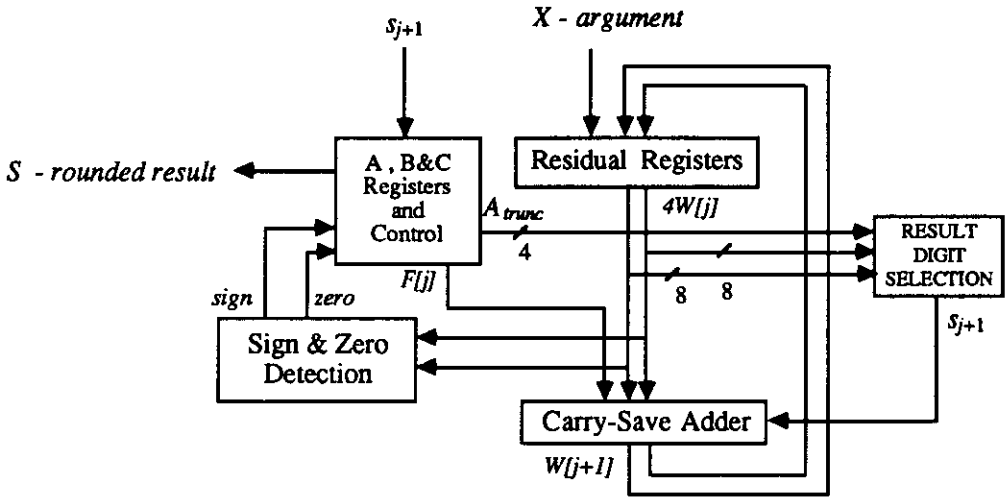

Figure 9. Block diagram of the square root scheme (mantissa part)