A FULL CHARACTERIZATION OF PSEUDOGRAPHOIDS IN TERMS OF FAMILIES OF UNDIRECTED GRAPHS

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ABSTRACT

Given a set of independencies closed under the pseudographoid axioms, an algorithm for constructing a set of undirected graphs perfectly representing the given set of independencies is provided and its correctness is proved. Based on this algorithm, a full characterization of pseudographoids in terms of undirected graphs is given. A possible extension for full graphoids is investigated and some open problems are proposed. The algorithms introduced are illustrated with many examples. The main result of this report properly generalizes the I-mapness theorem in the [Pearl & Paz, 1986] report.

1. Introduction

Given a basis $B = \{t_1 \cdots t_k\}$ consisting of triples $t_i = (x, z, y)$ where x, y, z are disjoint sets of nodes over a universal set of nodes N.

The triples represent independencies: The triple (x, z, y) represents the statement that "knowing z renders x and y independent.

Assume:

1. $k = |B| = \text{poly}(|N|)$ (i.e. the number of triples in B is polynomial in the number of nodes).

2. $\mathbf{cl}_{pg}(B)$ is the closure of B under the pseudographoid axioms: (zy denotes the union of

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the sets z and y and similarly for zw, etc.).

s. **Symmetry** : $(x, z, y) \rightarrow (y, z, x)$

d. **Decomposition** : $(x, z, yw) \rightarrow (x, z, y)$ & (x, z, w)

in. **Intersection** : (x, zy, w) & $(x, zw, y) \rightarrow (x, z, wy)$

u. **Weak union** : $(x, z, wy) \rightarrow (x, zw, y)$

Our goal is to develop an algorithm for representing $\mathbf{cl}_{pg}(B)$ by a set of graphs over N such that $t \in \mathbf{cl}_{pg}(B)$ **iff** z is a cutset between x and y in some graph in the set of graphs.

2. Some Definitions:

Represent the set of nodes by a set of vertices V. V_i denotes a subset of V.

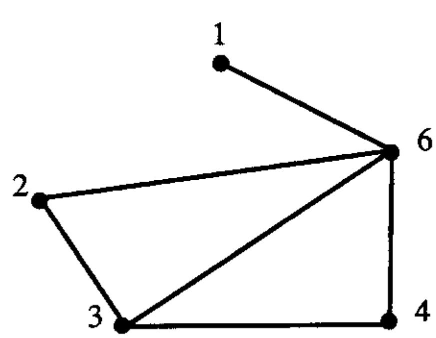
$G_i(V_i)$ is a graph over V_i.

If α, β, are elements of V_i then "(α, β) is a *nonedge* of G_i" means that (α, β) is not in the edge set of G_i.

For $V_j \underset{\neq}{\subset} V_i$

define (α, β) as a *nonedge* of G_i mod V_j if α is not connected to β in $G_i(V_i/V_j)$ i.e. removing the vertices V_j and the incident edges from $G_i(V_i)$ will render α and β disconnected.

Example.



$G_i(V_i)$

$V = \{1, 2, 3, 4, 5, 6\}$

Figure 1

G_i (V_i) is the above graph. (α, β) = (1,2) is a nonedge of G_i and is a nonedge of G_i mod {4, 6} but (α, β) = (1, 2) is *not* a nonedge of G_i mod {4}.

Remarks:

- (α, β) is a nonedge of G_i implies that (α, β) is a nonedge of G_i mod $\{V_i - \alpha - \beta\}$
- (α, β) is a nonedge of G_i mod V_j , ($V_j \subseteq V_i - \alpha - \beta$) implies that ($\alpha$, β) is a nonedge of G_i. The contrary is not always true as the above example shows.
- V_j ($\subset V_k \subset V_i$ and (α, β) is a nonedge of G_i mod V_j implies that (α, β) is a nonedge of G_i mod V_k

Define: Given G_i (V_i), G_j (V_j) $V_j \subseteq V_i$. G_j is *implied* by G_i if every nonedge (α, β) of G_j is a nonedge of G_i mod ($V_j - \alpha - \beta$).

Example.

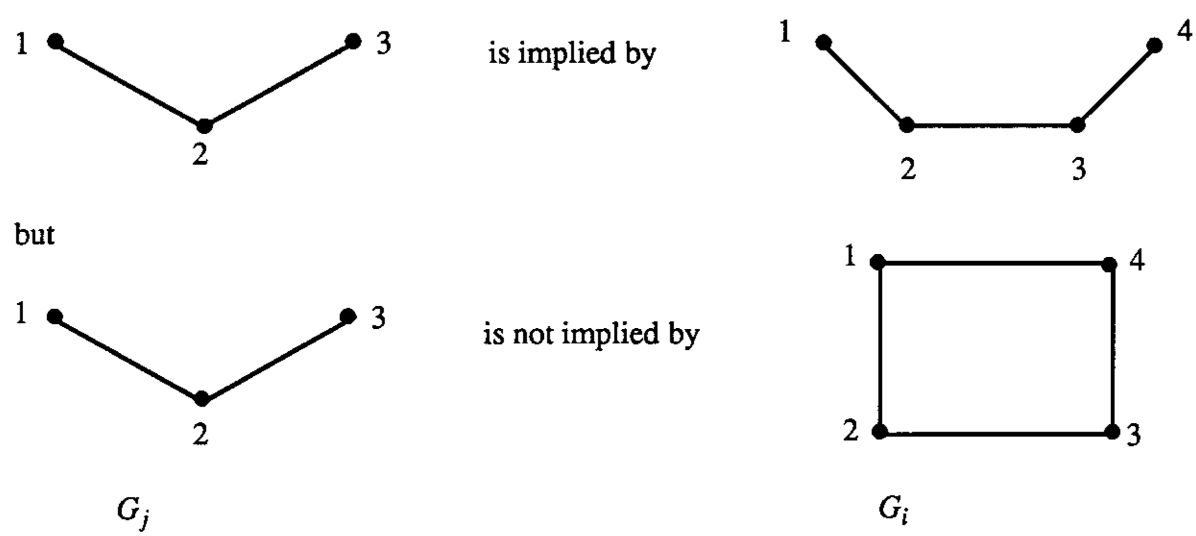


Figure 2

as (1, 3) is not a nonedge of G_j mod 2 in the second case.

- Checking whether G_j is implied by G_i is polynomial (in the number of vertices) as is

- Any *implied* G_j can be removed from the set of graphs representing a relation and *complete* graphs are superfluous in such a set.

3. The ⊗ operation on graphs.

Given 2 graphs $G_i(V_i)$ and $G_j(V_j)$ let $V_k = V_i \cap V_j$ and assume $V_k \neq \phi$. Define the graph $G_k(V_k) = G_i \otimes G_j$ as follows:

1. Every edge of G_i and G_j over V_k is an edge of $G_k(V_k)$
2. Every pair (α, β) over V_k which is a nonedge of G_i mod $(V_k - \alpha - \beta)$ *or* is a nonedge of G_j mod $(V_k - \alpha - \beta)$ is a nonedge of G_k
3. Every pair (α, β) to which 1 or 2 above does not apply is an edge of G_k

Example 1.

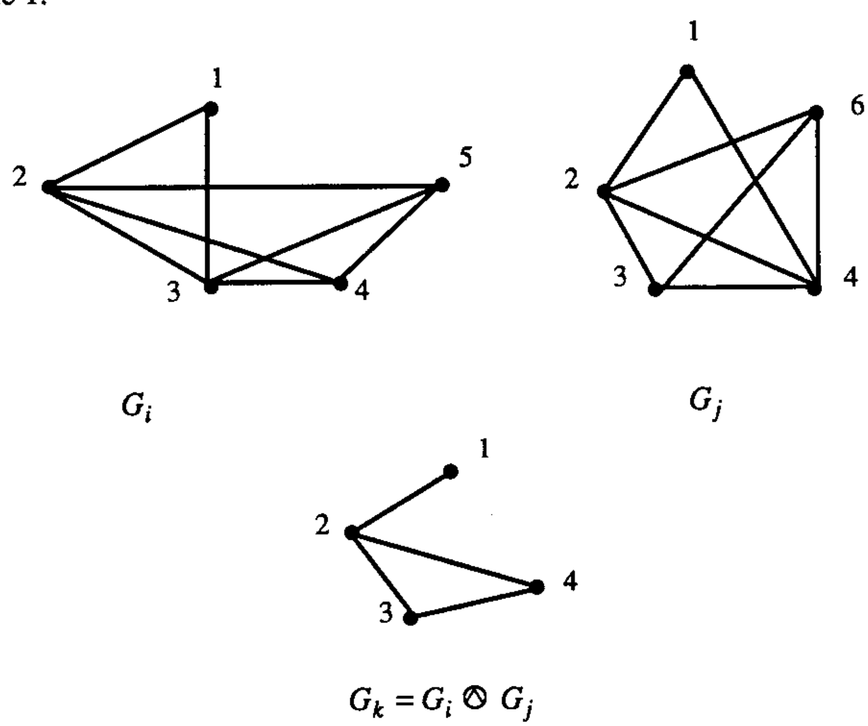
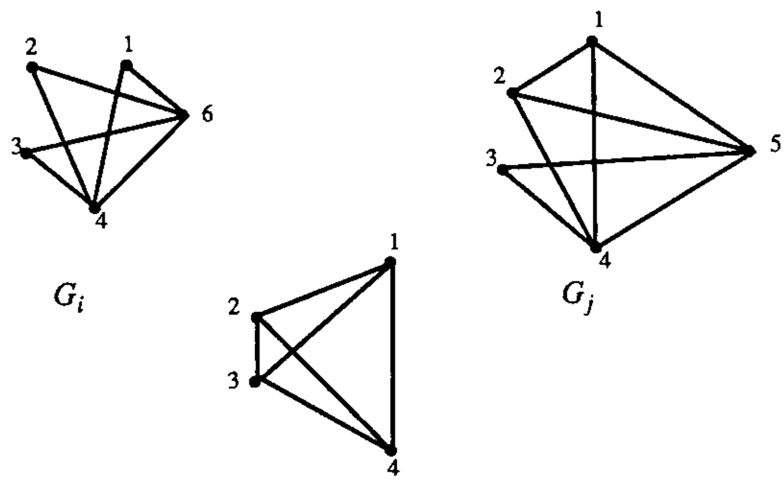


Figure 3

G_k is not implied by either G_i nor G_j. e.g. (1,2,34), (1,2,3) and (1,2,4) represented in G_k are not represented in G_i nor in G_j.

Example 2.

Example 2.



$G_k = G_i \obslash G_j$

Figure 4

In example 2, (1,4), (2,4), (3,4) are edges of G_k by the first rule. (1,2) is an edge of G_k since (1,2) is not a nonedge of G_i mod {3,4} and a similar situation exists for (2,3) and (3,1).

Here the G_k graph is complete and is therefore superfluous.

Remark: The construction of G_k is polynomial (in the number of vertices) as is easily seen. The above definition implies the following:

Lemma 1: If the independencies represented in G_i and G_j belong to $\mathbf{cl}_{pg}(B)$, then the independencies represented in $G_i \obslash G_j = G_k$ belong to $\mathbf{cl}_{pg}(B)$.

Proof: Any independency represented in G_k of the form $(\alpha, V_k - \alpha - \beta, \beta)$ is either an independency represented in G_i or in G_j (see rule 2 in the construction). The proof can now be completed by descending induction and is similar to the proof of Theorem 1 in [Paz & Pearl, 1986].*

Lemma 2. If $V_j \subseteq V_i$ then $G_k = G_i \obslash G_j$ is a graph over V_j and is a subgraph of G_j.

* In the [Paz & Pearl, 1986] paper the system studied was termed "graphoid." As the theory developed (after the appearance of that paper), the nomenclature was expanded and the term graphoid is now reserved for a more restricted system obeying an additional axiom.

Proof: If (α, β) is a nonedge of G_j then it is a nonedge of G_j mod $G_j - \alpha - \beta$ and therefore (by rule 2) it is a nonedge of G_k. Thus the set of edges of G_k is a subset of the set of edges of G_j. Also, $V_k = V_j \cap V_i = V_j$ which completes the proof.

Corollary 3. If $G_k = G_i \otimes G_j$ and $V_j \subseteq V_i$ then G_k implies G_j and G_j can be discarded.

Corollary 4. If $G_k = G_i \otimes G_j$ and $V_j = V_i$ then G_k implies both G_j and G_i which can be discarded.

4. Similar Triples and Their Representation

Definition: Two triples t_1 and t_2 over a set V are *similar* if the set of vertices represented in t_1 and the set of vertices represented in t_2 are equal (the set of vertices *represented* in $t = (x, z, y)$ is $x \cup z \cup y$). Similarity is an equivalence relation as is easy to see.

Lemma 5. Let $T = \{t_1, \ldots t_k\}$ be a set of similar triples, then $\mathbf{cl}_{pg}(T)$ can be perfectly represented by a single graph.

Proof: Let V_i be the set of vertices represented in every triple in T. Construct G_i, a graph over V_i by removing from the complete graph over V_i every edge (α, β), such that $\alpha \in x_i$, $\beta \in y_i$ for some triple $t_i = (x_i, z_i, y_i) \in T$, and only those edges. The constructed graph perfectly represents T. This can be shown as follows:

Let $t_1, t_2 \ldots t_k, t_{k+1} \ldots t_m$ be a list of all triples in $\mathbf{cl}\,(T)$ ordered in a way such that any t_i, $i > k$ is derived from previous triples in the list by one of the axioms. We can show now that every triple in the list is represented in the graph by finite induction: The statement is clearly true for $t_1 \cdots t_k$ and the truth for $j > k$ is implied by the fact that graphs satisfy the 4 axioms, and by the induction hypothesis. The other direction of the proof follows from Theorem 1 in [Pearl & Paz, 1986].

5. The Set cl(W) and its Construction

Given a set of graphs

$$G = \left\{ G_i(V_i) : V_i \subseteq V \right\}$$

Let W be the set

$$W = \left\{ V_i : G_i(V_i) \in G \right\}$$

Let **cl** (W) be the closure of W with respect to the operation of intersection. **cl** (W) is a lattice with inclusion as a partial order.

If $V_i \in$ **cl** (W), we define the rank of V_i, $r(V_i)$ as follows:

If $V_i \subset V_{j1} \subset V_{j2} \subset \cdots \subset V_{jt}$ is a chain of maximal length in **cl** (W) then $r(V_i) = t$. In particular if V_i is a maximal element in **cl** (W), then $r(V_i) = \phi$.

It follows from the definitions that:

1. For any $V_i \in$ **cl** (W) $r(V_i) \leq | V |$ ($| V |$ is the number of nodes in V)

2. $V_i, V_j \in$ **cl** $(W) \Rightarrow V_i \cap V_j \in$ **cl** (W).

6. The Set cl (G) and its Construction

Based on **cl** (W) construct inductively **cl** (G) (the closure of G) as follows.

1. Set **cl** $(G) = \phi$

2. For all maximal $V_i \in$ **cl** (W): combine via the $\otimes$ operation all the G_i's in G whose vertex set is equal to V_i. Let $\overline{G}_i(V_i)$ be the resulting graph. If $\overline{G}_i(V_i)$ is not the complete graph then set

$$\textbf{cl}\ (G) = \textbf{cl}\ (G) \cup \overline{G}_i(V_i)$$

3. For $t = 1, 2, \ldots r_{\max}$ ($r_{\max}$ is the maximal rank of an element in **cl** (W) and is $\leq | V |$):

   For all V_i with $r(V_i) = t$ in **cl** (W): Let $V_{j_1}, \cdots, V_{j_s}$ be all the immediate predecessors of V_i in **cl** (W) (V_j is an immediate predecessor of V_i if $V_i \subset V_j$ and for no V_p, $V_i \subset V_p \subset V_j$).

Combine via the $\otimes$ operation all the G_i's in **cl** (G) whose vertex set is equal to $V_{j_1}, \cdots V_{j_s}$ (if any), together with all the G_i's in G whose vertex set is equal to V_i.

Let $\overline{G}_i(V_i)$ be the resulting graph. If $\overline{G}_i(V_i)$ is not the complete graph and in addition $\overline{G}_i(V_i)$ is not implied by any of the graphs $\overline{G}_{j_1}(V_{j_1}), \cdots, \overline{G}_{j_s}(V_{j_s})$ already in **cl** (G) then set

$$\mathbf{cl}\,(G) = \mathbf{cl}\,(G) \cup \overline{G}(V_i)$$

end of construction □

Remarks

1. It follows from the definition that the $\otimes$ operation is commutative, associative and idempotent, (i.e. $G \otimes G = G$) and therefore the above construction is well defined and there is a 1 - 1 correspondence between **cl** (G) and **cl** (W) (also between G and W).

2. It follows from Lemma 4 that $\overline{G}_i(V_i)$ implies all the $G_i(V_i)$ in G with the same vertex set V_i.

3. If the number of elements in **cl** (W) is polynomial in $|V|$ then the whole construction is polynomial in $|V|$. (This is the case e.g. if the number of elements in G (or in W) is logarithmic in $|V|$).

   We are now ready to present the Algorithm.

7. A Perfect Representation of a Pseudographoid by a Set of Graphs

Consider the following algorithm.

1. Input $T = \{t_1 \cdots t_m\}$ over a set of nodes N, and the 4 pseudographoid axioms. Assume $m = f(n)$, $n = |N|$ (for further reference).

2. Separate T into equivalence classes $T = \{T_1, T_2 \cdots T_s\}$ of triples

3. For each T_i construct a graph G_i, as in Lemma 5, which perfectly represents T_i.

4. Let $G = \{G_1, \cdots G_s\}$ the set of graphs constructed in the previous step. Construct **cl** (G) as described in Section 6 above.

We now have the following:

Theorem 6: The set of graphs **cl** (G) perfectly represents the pseudographoid T.

Proof: Every cutset in one of the graphs in **cl** (G) represents an independency in **cl** $_{pg}(T)$. This follows from the construction and from Lemma 1. We need therefore to show only that any independency in **cl** (T) is represented in some graph in **cl** (G).

We show first that the independencies in T are represented:

By step 3 of the algorithm every independency in T is represented in some G_i. By step 2 of the algorithm, no two G_i's in G have the same vertex set. Therefore, by step 2 in the construction of **cl** (G), all the graphs G_i corresponding to maximal sets of vertices are included in **cl** (G). If a G_i in G corresponds to a nonmaximal set of vertices then, when it is processed in step 3 of the **cl** (G) algorithm, it is combined with graphs in **cl** (G) (via the $\otimes$ operation), whose vertex sets include the vertex set of G_i as a subset. Therefore, the resulting $\overline{G}_i(V_i)$ implies G_i (corollary 3), and thus all the independencies in G_i are represented in $\overline{G}_i$ which is included in **cl** (G). We have thus shown that all independencies in T are represented in some $G_i \in$ **cl** (G).

To complete the proof, consider a triple t which is not in T, but is derived from triples in T by a chain of derivations using the axioms (**s, d, in, u**).

Let

$$t_1, t_2 \cdots t_{p-1}, t_p = t$$

be the derivation chain.

For any t_i in the chain, either $t_i \in T$ or t_i is derived from T_j, $j < i$ by one of the axioms **s**, **d**, or **u**, or t_i is derived from previous triples t_j, t_k $j, k < i$ by the **in**-axiom. Proceed now by induction to show that all triplets in the chain are represented.

Basis: t_1 must be in T and is therefore represented by the first part of the proof.

Step. Assume t_s is represented for all $s < i$. If t_i is derived from t_j, $j < i$ by axioms **s**,**d**,or **u**, then t_j, being represented in some graph (induction hypothesis) implies that t_i is represented in the same graph (graphs satisfy above 3 axioms with independencies corresponding to cutsets).

If t_i is derived from t_j and t_k $j, k < i$, by the **in**-axiom then (induction hypothesis) t_j and t_k are represented in some graph in **cl** (G). If t_j and t_k are represented in the same graph, then we are done (graphs satisfy the **in**-axiom). We will show that this is always the case.

Let t_j be represented in the graph $G_j(V_j)$

Let t_k be represented in the graph $G_k((V_k)$

and Let $V_j \cap V_k = V_i$

Set $t_j = (x, zw, y)$ and let α, β be any vertices $\alpha \in x$, $\beta \in y$. Then $<\alpha \mid zw \mid \beta>_{G_j}$, or

(α, β) is a nonedge of G_j mod $\{z \cup w\}$.

If $G_p(V_p)$ is a graph in **cl** (G) such that $\{x \cup z \cup w \cup y\} \subset V_p$ and $G_p = G_j \oslash G_s$ for some graph G_s in **cl** (G) then, by the definition of $\oslash (\alpha, \beta)$ is a nonedge of G_p and since α, β are arbitrary vertices in x and y, we know that t_j is represented in $G_p(V_p)$.

Set $V_i = V_j \cap V_k$. There is a graph $G_i(V_i)$ in **cl** (G) which is derived from V_j via a sequence of $\oslash$ operations in which each intermediary graph G_p includes in its vertex the set $\{x \cup z \cup w \cup y\}$. Therefore t_j is represented in each intermediary G_p and also in G_i. Similarly t_k is represented in G_i and the proof is now complete.

8. An Example

Let $T = \{ (27, 1, 35689), (12348, 67, 9), (127, 8, 369), (8, 27, 469), (6, 4, 28), (68, 4, 9) \}$.

The resulting **cl** (W) is given below, ordered by inclusion:

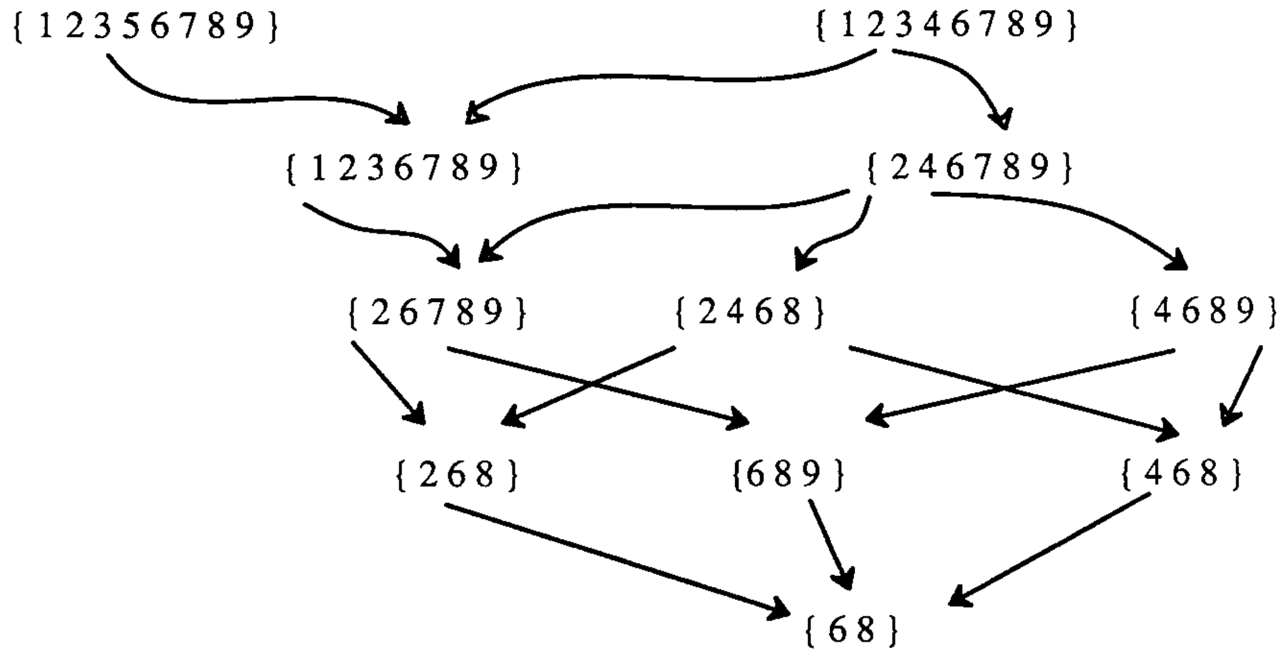
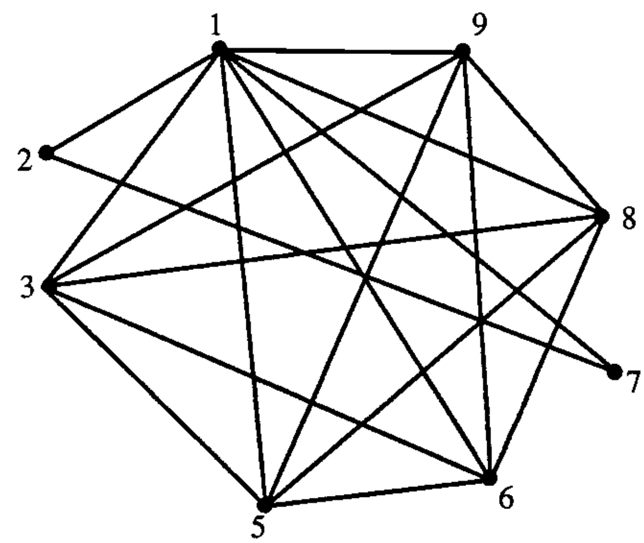


Figure 5

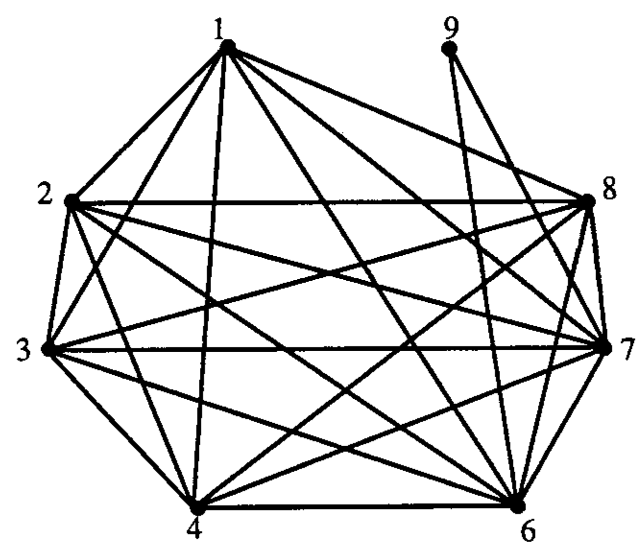
Starting from the 6 graphs defined by T (a different graph for every triple) the **cl** (G) can be constructed according to the given algorithm.

The resulting set of graphs is shown below. The graphs resulting from the algorithm and corresponding to the sets *{268}*, *{689}*, *{468}* and *{68}* are implied by previous graphs in the family and are therefore omitted.

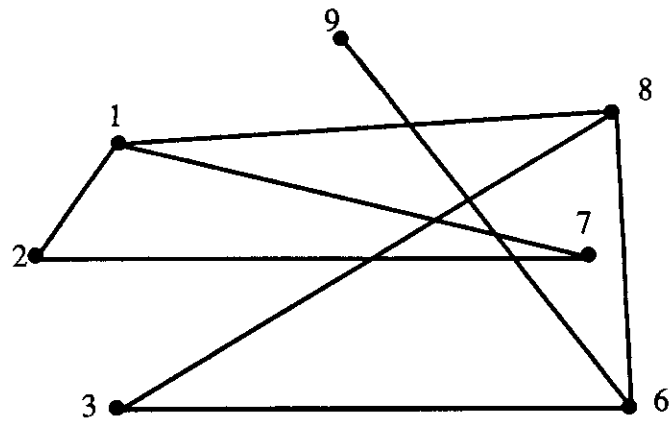
SET GRAPHS



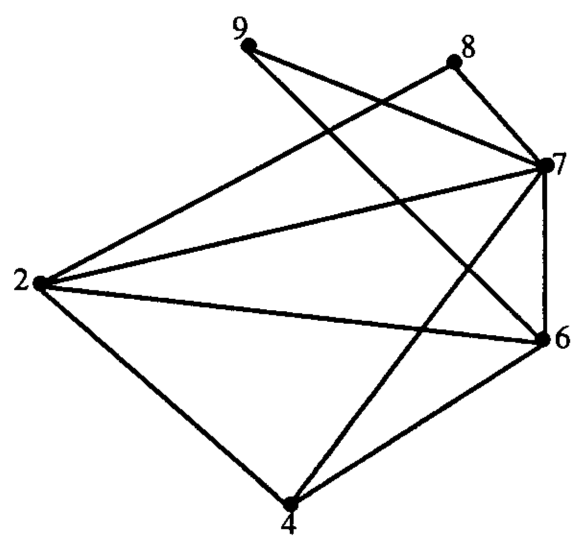
Graph G_1 : Corresponds to the set {1 2 3 5 6 7 8 9}



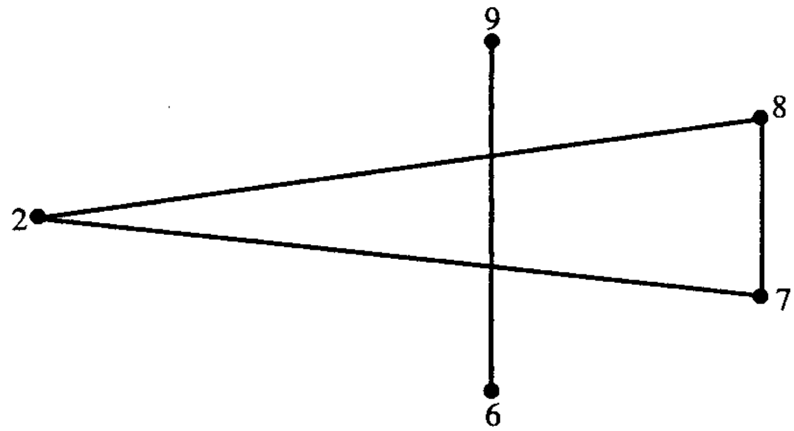
Graph G_2 : Corresponds to the set {1 2 3 4 6 7 8 9}



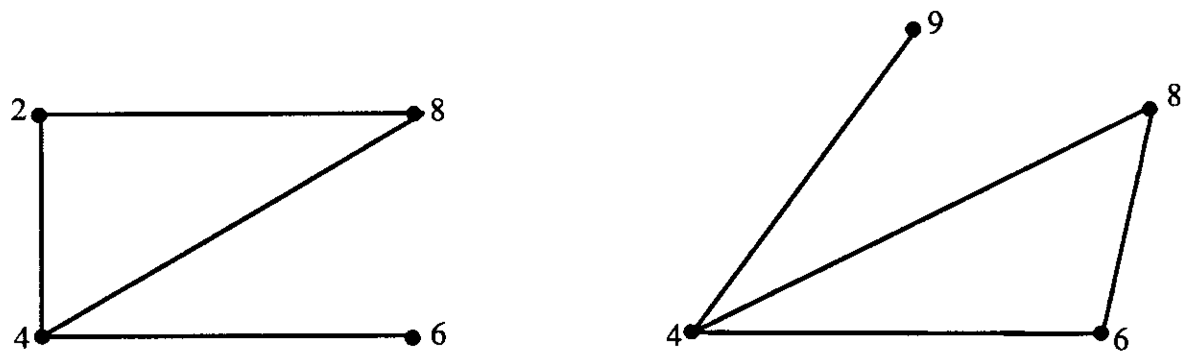
Graph G_3 : Corresponds to the set {1 2 3 6 7 8 9}
G_3 is derived from $G_1 \otimes G_2$ and the third triple in T.



Graph G_4 : Derived from G_2 and the fourth triple in T



Graph G_5 ; $G_5 = G_3 \otimes G_4$



Graph G_6 and G_7 both derived from G_4 and the last 2 triplets in T

9. A Complete Characterization of Pseudographoids

The construction in Section 7 provides a complete characterization of pseudographoids. The characterization is stated in the following:

Theorem 7. For any given pseudographoid T ('given' as specified in Section 1), one can construct a set of graphs closed under the $\otimes$ operation and such that $t = (x, z, y) \in T$, if and only if z is a cutset between sets of vertices x and y in some graph G_i in the set of graphs.

Conversely, given a set of graphs $G = \{G_i(V_i) : V_i \subseteq V\}$ closed under the $\oslash$ operations, the set of cutset triples $\{<x|z|y>_{G_i} : G_i \in G\}$ is a pseudographoid (i.e. the set satisfies the pseudographoid axioms).

The proof of this theorem follows directly from the construction in Section 7.

Corollary 8: Every non-extreme probabilistic model can be perfectly represented by a family of graphs closed under the $\oslash$ operation.

Notice that the converse is not necessarily true e.g. if the set of independencies is not closed under contraction. Notice also that for single graphs the opposite holds i.e. for every single undirected graph there exists a probabilistic model perfectly representing its set of independencies, but not the other way around [Geiger, Section 4, 1987].

10. Some Remarks.

1. In the [Paz & Pearl, 1986] paper, the concept of I-mapness has been introduced for pseudographoids. It was shown there that for every pseudographoid T a minimal (single) I-map graph G_T can be constructed such that any cutset in G_T represents some independency in T, but not the other way around.

   The theory developed in this report is a proper extension of that theory. e.g. for the example in Section 8, the minimal I-map would be the complete graph on the vertices $\{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\}$ which is a trivial and vacuous I-map of the given relation. Here we were able to perfectly represent that relation by 7 graphs.

2. The number of graphs in the family of graphs constructed in Section 7 for a given pseudographoid depends on the number of elements in the set $\mathbf{cl}(W)$ where $W = \{V_i : G_i(V_i) \in G\}$ and closure is with regard to the intersection of sets (see Section 5). If that number is polynomial in $|N|$ (see Section 1), then a polynomial algorithm will result for the membership problem (i.e. finding for any given triple t whether $t \in \mathbf{cl}_{pg}(T)$).

   The structure of $\mathbf{cl}(W)$ is very simple and straightforward, and can easily be determined before any further computations (see Section 5).

3. As the number of subsets of a given set is exponential (in the cardinality of the source set), it might be the case that the cardinality of the resulting family of graphs, perfectly representing a pseudographoid given by a polynomial base, is exponential. If this is the case we might still use the algorithm in order to get as good an approximation (in the I-

mapness sense), as time or space allows. It is clear from the definition of the algorithm that every time an iteration is performed successfully, a better I-map approximation is achieved.

11. Possible Extensions.

The construction described in Section 7 can possibly be extended to full graphoids, i.e. systems which obey, in addition to 4 pseudographoid axioms, the contraction axiom

$$\mathbf{c} : (x, zy, w) \ \& \ (x, z, y) \rightarrow (x, z, yw).$$

We believe that the algorithm presented below will ultimately provide such an extension, but additional investigation is need in order to provide a complete proof of our claim.

Contraction Algorithm for 2 graphs

Given 2 graphs denoted by R (red) and B (blue), for convenience.
Let V_{RB} and E_{RB} be the set of vertices and edges, correspondingly, common to both graphs, let V_R, E_R, V_B, E_B denote the monochromatic vertices and edges correspondingly.

Assume that (x, zy, w) is represented in R and (x, z, y) is represented in B. If contraction applies to these two premises then there must be two vertices in V_{RB}, say a and b, such that

i. $a \in x$, $b \in y$

ii. (a, b) is an edge in E_R

iii. (a, b) is a nonedge in B mod $V_{BR} - a - b$.

Explanation: All connections between x and w in R must pass through z and y and some must directly connect through y. Otherwise they connect directly through z only and (x, z, yw) is already in R. This explains ii. Now, since (x, z, y) is represented in B and $x, z, y \in V_{RB}$, we must be able to separate x from y by $V_{RB} - x - y$. This leads to the first step in the algorithm.

Step 1. For R, list all pairs (a, b), $a, b \in V_{BR}$, satisfying condition ii. and iii. above, and do the same for B (i.e. list pairs $(a, b) : (a, b)$ is an edge in E_B and is a nonedge in R mod $V_{BR} - a - b$). If both lists are empty abort. No new independencies are implied. The next step will be explained in the sequel.

Step 2. For R, for any listed pair (a, b) do the following:

2.1 If either $(a, V_{BR} - a, V_R - V_{BR})$ or $(b, V_{BR} - b, V_R - V_{BR})$ is represented in R, then disconnect a from b in R.

2.2 If $(a, V_{BR} - a, U_1)$ and/or $(b, V_{BR} - b, U_2)$ is/are represented in R for some nonempty set(s) $U_1, U_2 \subsetneq V_R - V_{BR}$ then

2.2.1 If only on of the above independencies is represented, say the first, construct a new graph which is the subgraph of R on the vertices $V_{BR} \cup U_1$ and disconnect a from b in it.

2.2.2 If both independencies are represented in R and $(V_{BR} \cup U_1) \subset (V_{BR} \cup U_2)$ or $(V_{BR} \cup U_2) \subset (V_{BR} \cup U_1)$ then proceed as in 2.2.1 with the subgraph of R constructed over the largest subset of the sets $V_{BR} \cup U_1$ and $V_{BR} \cup U_2$.

2.2.3 If both independencies are represented and the condition in 2.2.2 does not hold then construct 2 subgraphs of R, one over the subset of vertices $V_{BR} \cup U_1$, and the other over the subset of vertices $V_{BR} \cup U_2$, and disconnect a from b in those subgraphs.

3. If neither condition 2.1 nor condition 2.2 hold for the pair (a, b) ignore that pair and proceed to the next pair.

Apply the same procedure to B.

Explanation:

If the condition in 2.1 holds with $(a, V_{BR} - a, V_R - V_{BR})$ say, then, since (a, b) satisfies condition iii. of the listed pairs we have also that $(a, V_{BR} - a - b, b)$ is represented. We must now represent the implied new independency $(a, V_{BR} - c - b, (V_R - V_{BR}) \cup b)$. This is achieved by disconnecting a from b in R. If one of the conditions 2.2 holds then disconnecting a from b in R would create an independency of the form $(a, V_{BR} - a - b, (V_R - V_{BR}) \cup b)$, which is *not* the independency implied by the contraction in this case. New graphs must therefore be constructed, subgraphs of R (to preserve the closure under intersection) in which the new independencies will be represented. If an independency in B of the form (a, z, b) is to be combined with an independency in R to initiate a contraction, then that independency must be represented in B modulo V_{BR} and, due to the fact that graphs are closed under strong union we must also have that $(a, V_{BR} - a - b, b)$ is represented. Therefore, it is reasonable to assume that the procedure will take care of all the contractions implied by the two graphs. As long as there is an independency implied by contraction and not represented, some (a, b) pair will show in Step 1 of the algorithm which will be disconnected (in some graph) when the algorithm is completed. The algorithm is illustrated by examples in the next section.

Remarks.

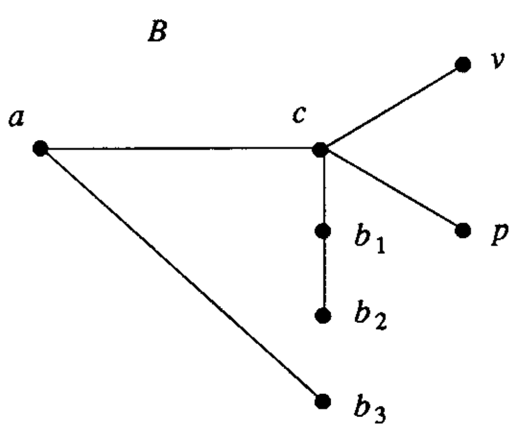
a. We assume that R and B belong to a family of graphs closed under the $\oslash$ operation and the algorithm should be applied for all such pairs of graphs. The two algorithms (the one in Section 7 and this one) should be applied in repeated succession until a family of graphs closed under both intersection and contraction is achieved.

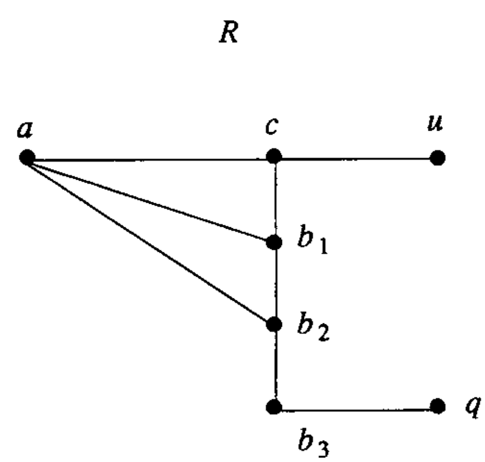
b. If the set of vertices of vertices of one of the graphs R and B is a subset of the other set of vertices, say $V_B \subset V_R$, then Steps 1 and 2 should be applied to the graph R only. This follows from the following observations. Assuming that $B = R \oslash B$ (both graphs belong to a set closed under the $\oslash$ operation), any edge of B (a, b) which is a nonedge of R has the property that (a, b) is not a nonedge of R mod $(V_{BR} - a - b)$ and therefore the edge (a, b) in B does not satisfy the condition iii. of the listed edges for B.

c. While combining 2 particular graphs is polynomial, the whole algorithm might be exponential. Still it is sure to terminate and we believe that it is correct (no independency is destroyed since no edges are added and edges are removed creating new independencies as long as this is possible. But the number of graphs on a fixed number of vertices is finite). Moreover one does not have to carry the algorithm to the end, and each time an iteration is successfully applied, a better approximation is achieved.

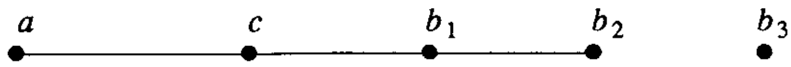
12. Examples.

1. Let B and R be the graphs below





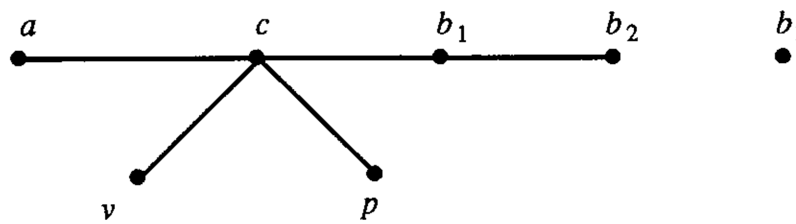
Assuming that the graphs belong to a family of graphs closed under the $\oslash$ operation, the family must contain also the following graph:



$= B \otimes R$

Step 1 of the contraction algorithm for B and R will result in the following lists.

For $B : (a, b_3)$. For $R : (a, b_1), (a, b_2), (b_2 b_3)$. Applying Step 2 to B we get that $(a, c\ b_1 b_2 b_3, v\ p)$ is represented in B (Sept 2.1). Therefore we must disconnect a from b_3 in B resulting in the following new B.



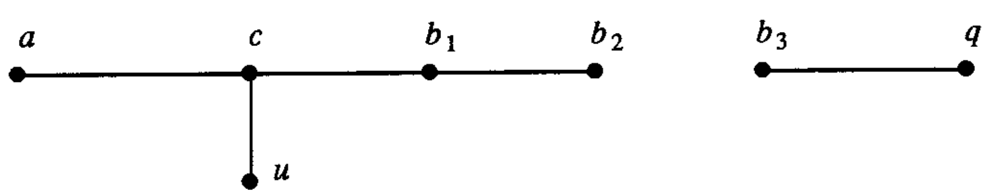
Applying Step 2 to R we get

For the pairs (a, b_1) and (a, b_2):

$(a, c\ b_1 b_2 b_3, u\ q)$ is represented (Step 2.1). We must therefore disconnect a from b_1 and b_2.

For the pair $(b_2 b_3)$:

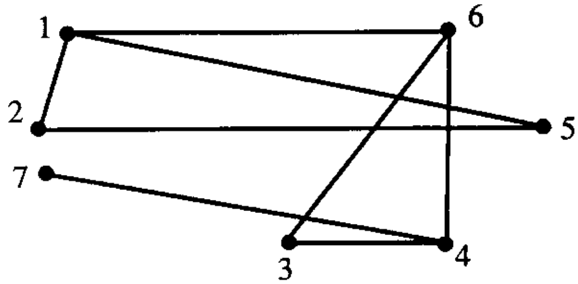
$(b_3, b_2 b_1\ c\ a\ , u)$ and $(b_2, b_3, b_1, c\ a, u\ q)$. Condition 2.1 is satisfied for the second pair and therefore we must disconnect b_2 from b_3 The resulting new R (after all three edges have been disconnected) is



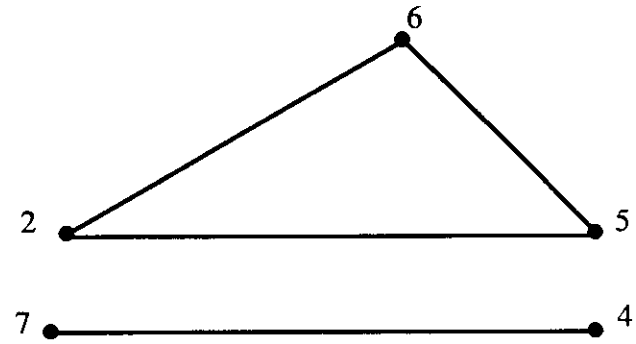
Notice that the $\otimes$ operation on the new graphs will result in the same third graph we had before. The set is now closed under both intersection and contraction.

2. Let B and R be the graphs below

B



R



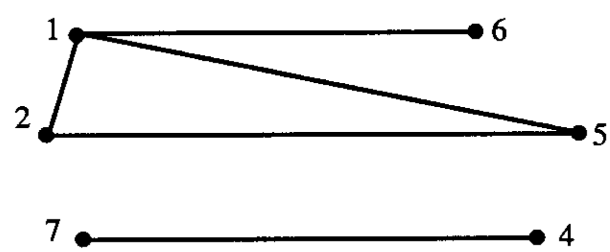
Since $V_R \subset V_B$ and $R = B \otimes R$ (the reader is urged to check), we need consider only B for the steps of the algorithm.

There is only one edge of B produced in Step 1, namely (6,4).

Applying Step 2 to this pair we get

(6, 2 5 7 4, ϕ) and (4, 2 5 6 7, 1)

The case here is 2.2.1. We must create a subgraph of B by eliminating the vertex 3 from it, and incident edges, and disconnect in it edge 6 from 4, resulting in



The two original graphs together with this new one constitute the closure of the original two graphs under both intersection and contraction.

13. Problems for Further Study

1. Provide a full formal proof of the algorithm is Section 11 and provide a full characterization of graphoids in terms of undirected graphs.

2. Characterize families of UGs (undirected graphs) of polynomial size, representing

pseudographoids or graphoids. Find and study useful or interesting such families.

3. [Pearl & Verma, 1987] have studied the representation of Independencies in Directed Acyclic Graphs (DAGS). If is known that both directed and undirected graphs obey all the 5 axioms of graphoids. What happens if a family of DAGs (instead of a single DAG) is considered as a possible model for representing an independency relation?

4. Is it possible to represent a polynomial size set of DAGs, closed under the graphoid axioms, by a polynomial size set of UGs closed under the same axioms? Is the converse possible? What about a single DAG (UG) versus a family of UGs (DAGs)?

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