THE NON-AXIOMATIZABILITY OF DEPENDENCIES IN DIRECTED ACYCLIC GRAPHS

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1. ABSTRACT

Recently, few models were proposed to capture the notion of relevance. Their main component consists of a mechanism to assign truth values to a 3-place relation I(x,z,y) where x,y,z are three non-intersecting sets of elements (e.g attributes or variables), and I(x,z,y) stands for the statement: "Knowing z renders x irrelevant to y." Among these models one can find the Probabilistic dependency model ([1]), the Undirected Graph (Pearl & Paz [2]), Directed Acyclic Graph (Pearl & Verma [3]) and Hybrid Acyclic Graph (Verma [4]). An important tool in investigating these models and their expressive power is a complete set of axioms. Such a set was found so far only for the Undirected Graph (UG) model. In this paper, we show that there is no finite complete set of Horn axioms for the Directed Acyclic Graph (DAG) models. Moreover, we compare our result to a similar result in the Embedded Multi Valued Dependency (EMVD) model of relational databases established by Parker and Parsaye in 1980 ([5]), and independently by Sagiv and Walecka in 1982 ([6]). We point out that a stronger version of their incompleteness theorem can be stated for EMVDs. However, a similar extension for DAGs has not been fully established so far.

The paper is organized as follows. Section 2 reviews some previous work and the necessary terminology concerning dependency theory and the DAG model in particular. Section 3 presents the construction which shows that there is no finite complete set of Horn axioms for DAGs. Section 4 discusses extensions to the incompleteness theorem. Finally, section 5 summarizes the results and outlines the relation between dependency models and relational database theory.

2. DEPENDENCY MODELS

To understand the need of dependency models it is most adequate to quote Pearl and Paz as follows: "Any system that reasons about knowledge and beliefs must make use of information about relevancies. If we have acquired a body of knowledge z and now wish to asses the truth of proposition x, it is important to know whether it would be worthwhile to consult another proposition y, which is not in z. In other words before we consult y we need to know if its truth value can potentially generate new information relative to x, information not available from z. In probability theory, the notion of relevance is given precise quantitative underpinning using the device of conditional independencies a variable x is said to be independent of y given the information z if

$$P(x , y \mid z) = P(x \mid z) P(y \mid z).$$

However, it is rather unreasonable to expect people or machines to resort to numerical verification of equalities in order to extract relevance information. The ease and conviction with which people detect relevance relationships strongly suggests that such information is readily available from the organizational structure of human memory, and not from numerical values assigned to its components. Accordingly, it would be interesting to explore how assertions about relevance can be tested in various models of memory and in particular, whether such assertions can be derived by simple manipulations on graphs."

In probability the predicate I(x,z,y) *"Knowing z renders x irrelevant to y"* is naturally defined as:

$$I(x,z,y) \leftrightarrow P(x\,,\,y \mid z\,) = P(x \mid z\,)\,P(y \mid z\,).$$

The definition above suggests two obvious ways of answering the query " Is I(x,z,y) true ?". The first is to keep a list of all triplets (x,z,y) for which I(x,z,y) holds. This solution is too expensive in space. The second, would be to keep the full distribution function P, and test whether the inequality above holds. This is too expensive in time since, for most queries, the time required for an answer grows exponentially with the number of variables in the system. For these reasons the use of graphical representations was suggested.

The simplest model is the undirected graph model (UG). In UG models the relation I(x,z,y) holds iff the variables of the set z block all paths from the set x to the set y. It was shown ([1]) that the following axioms hold for all UG models :

(1.a) Symmetry $I(x,z,y) \rightarrow I(y,z,x)$

(1.b) Decomposition-Composition $I(x,z,yw) \leftrightarrow I(x,z,y)\ \&\ I(x,z,w)$

(1.c) Intersection $I(x,zw,y)\ \&\ I(x,zy,w) \rightarrow I(x,z,yw)$

(1.d) Strong union $I(x,z,y) \rightarrow I(x,zw,y)$

(1.e) Transitivity $I(x,z,y) \rightarrow I(x,z,\gamma)$ or $I(\gamma,z,y)$ where $\gamma \notin x \cup y \cup z$

where x,y,z are disjoint sets of nodes and xy is the union of the sets x,y.

For convenience we call a statement of the form I(x,z,y) an *independency statement*. The semantics associated with an axiom are that whenever a UG model obeys the antecedents of an axiom (after instantiation of the variables) it must also obey at least one of the independency statements of the disjunction on the right hand side of the axiom. We call I(x,z,y) a *single consequence* of a set of independency statements S, if every UG model that obeys all statements in S also obeys I(x,z,y).

A set of axioms **A** is *complete* if for every set of independency statements, all single consequences can actually be derived using only the axioms in **A**. The notion of completeness is important as it guarantees that, given a set of independency statements, one can derive all its single consequences. Such a process is desired when seeking a UG model that best matches a given set of independency statements.

The disjunctive consequences are a natural extension for single consequences. A *disjunctive consequence* of a set of independency statements S is a disjunction of independency statements for which at least one is valid in every model that obeys S. Wherever the context allows diambiguation, the term *consequence* is used for either a disjunctive consequence or a single consequence.

Axioms 1.a - 1.e were found to be complete ([2]), however they suffer from computational disadvantages since the transitivity axiom has a disjunction on the right hand side of the implication. To emphasize this property we refer to such axioms as *disjunctive axioms*. All other axioms have a single consequence and are said to be *Horn axioms*. The following example shows the computational disadvantages imposed by a non-Horn axiom (1.e). Moreover, it demonstrates that in some cases transitivity must be applied in order to add a single consequence (and not only a disjunctive consequence) to a given set of independency statements.

Example: Let S={ I(x,z,y) , I(z,wy,γ) , I(z,wx,γ) | γ is a single variable not in $x \cup y \cup z \cup w$} be a set of independency statements. We show that I(γ,w,z) is a consequence of Σ that can not be derived without applying transitivity.

Using (1.e) on I(x,z,y) we get I(x,z,γ) or I(γ,z,y). Assume I(x,z,γ). Then using (1.d) we get I(x,zw,γ). Adding the independency I(z,xw,γ) (which is in S) and using (1.c) yields the independency I(γ,w,xz) which using (1.b) yields I(γ,w,z). Now,on the other hand, assume I(γ,z,y). Then using (1.d) we get I(y,zw,γ). Adding the independency I(z,yw,γ) (which is in S) and using (1.c) yields the independency I(γ,w,yz) which using (1.b) yields I(γ,w,z). Thus, I(γ,w,z) is a single consequence of S, because it was derived by the axioms of UG's. It is left to show that I(γ,w,z) can not be derived from S without the transitivity axiom. This is done by finding the closure of S under the axioms 1.a-1.d and verifying that I(γ,w,z) is not in the closure. Indeed the closure is the set:
{ I(x,z,y) , I(z,wy,γ) , I(z,wx,γ) , I(x,zw,y) , I(x,zγ,y) , I(x,zwγ,y) , I(z,wyx,γ) and symmetric images }.

This example demonstrates two important issues. The first is that in order to apply a non-Horn axiom one needs to select each term in the disjunction separately, and for each selection reach a common consequence. Such a process is computationally expensive because for each application of transitivity in a derivation, two new statements need to be considered independently, and each might require the use of transitivity once again. The second is, can we establish a complete set of Horn axioms for the UG model by replacing the transitivity axiom with a finite set of Horn axioms, say,

$$I(x,z,y) \ \& \ I(z,wy,\gamma) \ \& \ I(z,wx,\gamma) \rightarrow I(\gamma,w,z).$$

We do not have the answer for this question; however, the example above shows the computational benefits of obtaining a complete set that consists solely of Horn axioms. The purpose of this paper is to show that while the task of finding a complete finite set of Horn axioms for the UG model is worthwhile, a similar task for the more complicated models (DAGs) can not be fruitful since such a set does not exist.

The weakness of the UG model lies in the restriction imposed by 1.e. This restriction prevents us from representing non-transitive relevance relations often found in probabilistic models as well as in common-sense reasoning. Therefore, let us consider the more refined model of Directed Acyclic Graphs (DAG). The DAG model, is defined by specifying how truth values are assigned to the predicates I(x,z,y) where x,z,y are three disjoint sets of nodes in a DAG. The motivation for this definition is beyond the scope of this paper and can be found in [1].

Definition:

a. Two arrows meeting head-to tail or tail-to-tail at node γ are said to be *blocked* by a set of vertices S if γ belongs to the set S.

b. Two arrows meeting head-to-head at node γ are *blocked* by S if neither γ nor any of its descendants is in S.

Definition:

a. An undirected path P in a DAG G is said to be *d-separated* by a subset of vertices if at least one pair of successive arrows along P is blocked by S.

b. Let x,y and S be three disjoint sets of vertices in a DAG G. S is said to *d-separate* x from y if all paths between x and y are d-separated by S. If such separation holds then I(x,S,y) is assigned the value true by this model.

Note that whenever we refer to DAGs, we assume the use of the d-separation criteria for I(x,z,y), even though other criterias might pop into mind.

In addition to 1.a-1.c, DAGs also satisfy the following properties [1]:

(2.d) Weak union $\quad I(x,z,wy) \rightarrow I(x,zw,y)$

(2.e) Weak transitivity $\quad I(x,z,y)\ \&\ I(x,z\cup\gamma,y) \rightarrow I(x,z,\gamma)$ or $I(\gamma,z,y)$ where $\gamma \notin x\cup y\cup z$

(2.f) Chordality $\quad I(\alpha,\gamma\delta,\beta)\ \&\ I(\gamma,\alpha\beta,\delta) \rightarrow I(\alpha,\gamma,\beta)$ or $I(\alpha,\delta,\beta)$ where α, β, γ and δ are distinct single elements.

However, a complete finite set of axioms was not found. In this paper we prove that a complete finite set of Horn axioms for DAGs does not exist. The proof that we present does not eliminate the existence of a finite complete set of disjunctive axioms for DAGs, however, we give some arguments suggesting that such a set does not exist.

3. NON AXIOMATIZABILITY OF DAGS

In this section we prove the incompleteness theorem. This theorem states that there is no finite complete set of Horn axioms for DAGs with the d-separation criteria. We use two lemmas in the proof. In the first lemma (3.1) we present a Horn axiom, called $R_0(n)$, with n antecedents, denoted S, and prove that $R_0(n)$ holds for all DAGs. In the second lemma (3.2) we list all single consequences of S (for $n \geq 7$) and prove that neither of them is a single consequence of any proper subset of S. The proof of the incompleteness theorem then follows using the following argument:

Define the *arity* of an axiom to be the number of antecedents of that axiom. Using this definition, lemma 3.2 states that for n≥ 7 the axiom $R_0(n)$ is irreducible to a chain of smaller arity Horn axioms. We proceed by contradiction. Assume A is a complete finite set of Horn axioms. Let k be the largest arity of all the axioms in A. Pick n=max(k+1,7). Consider the axiom $R_0(n)$. By lemma 3.2, this axiom is irreducible to a chain of smaller arity Horn axioms. All axioms in A have smaller arity then the arity of $R_0(n)$. Therefore, the consequence of $R_0(n)$ can not be derived from its antecedents by using only the axioms in A. This contradicts our assumption that A is complete.

We now prove the two lemmas.

Lemma 3.1: The axiom $R_0(n)$:

$$I(A_2,A_1,A_3)\ \&\ I(A_3,A_2,A_4)\ \&\ \cdots\ I(A_n,A_{n-1},A_{n+1})\ \&\ I(A_{n+1},A_n,A_1) \rightarrow I(A_{n+1},\emptyset,A_n \cup A_1)$$

holds in DAGs.

Proof (By contradiction): Assume $R_0(n)$ does not hold. Consider a DAG that obeys all antecedents of $R_0(n)$ and does not obey $I(A_{n+1},\emptyset,A_n \cup A_1)$. In this DAG, there exists a path between an element α_{n+1} of A_{n+1} and an element β of $A_n \cup A_1$, that is not d-separated by $\emptyset$. Clearly, this path does not contain a head-to-head node. Let P=(α_{n+1},β) be the shortest such path. Two cases need to be examined; β belongs to A_1 and β belongs to A_n.

Case 1: P=(α_{n+1},α_1) where α_1 belongs to A_1. The statement $I(A_{n+1},A_n,A_1)$ implies that A_n d-separates the path P. P does not contain a head-to-head node and therefore, in order to block two successive arrows, an element α_n of A_n must reside on P. Consider the path (α_{n+1},α_n). It does not contain a head-to-head node because P does not contain such node. Hence, (α_{n+1},α_n) is a path from A_{n+1} to $A_n \cup A_1$ that is not d-separated by $\emptyset$. This path contradicts our definition of P, because it is shorter then P.

Case 2: P=(α_{n+1},α_n) where α_n belongs to A_n. Using similar arguments, the statement $I(A_{n+1},A_{n-1},A_n)$ implies that an element α_{n-1} of A_{n-1} is on P. Consider the path (α_{n-1},α_n). This path has no head-to-head nodes, therefore the statement $I(A_{n-1},A_{n-2},A_n)$ implies that an element α_{n-2} of A_{n-2} must reside on (α_{n-1},α_n). Similarly, an element of each of the sets $A_{n-3}, A_{n-4}, \ldots, A_1$ must reside on P. Consider the path (α_{n+1},α_1) where α_1 is an element of A_1 that is on P. This path contradicts our definition of P because it is shorter then P. □

Lemma 3.2: Let S be all the antecedents of $R_0(n)$. Then the only non-trivial single consequences of S for n≥ 7 are: $I(A_{n+1},A_1,A_n)$, $I(A_{n+1},\emptyset,A_n)$, $I(A_{n+1},\emptyset,A_1)$, $I(A_{n+1},\emptyset,A_n \cup A_1)$, none of which is a consequence of any proper subset of S.

Proof: Let VALID be the set $\{ I(A_{n+1},A_1,A_n), I(A_{n+1},\emptyset,A_n), I(A_{n+1},\emptyset,A_1), I(A_{n+1},\emptyset,A_n \cup A_1) \}$. From lemma 3.1 we know that $I(A_{n+1},\emptyset,A_n \cup A_1)$ is a consequence of S and the other three statements in VALID can immediately be derived from it using decomposition (1.b) and the weak union (2.d) axioms.

Assume I(x,z,y) is an arbitrary non trivial independency statement not in **VALID**. We will show that I(x,z,y) is not a consequence of S by constructing a DAG that obeys S and does not obey I(x,z,y). Without loss of generality assume that the A_i's are singletons, $A_i=\{a_i\}$ i =1..n+1 and that these are all the nodes of the DAG (It is enough to contradict the axiom for any assignment of A_i's). Also w.l.o.g, x and y can be considered as singletons due to the decomposition axiom:

$$I(x_1,z,y_1 \cup y_2) \rightarrow I(x_1,z,y_1) \;\&\; I(x_1,z,y_2).$$

Namely, if $I(x_1,z,y_1)$ is not a possible consequence then neither is I(x,z,y) where x,y are any sets containing x_1 and y_1 respectively. For any assignment of x,y,z, our task is to construct a DAG that satisfies all the n antecedents of R_0(n) but violates I(x,z,y).

Assume $x=A_j$, $y=A_k$, j<k and examine the statement $I(A_j,z,A_k)$ for all z and for all possible values of j and k. We will say that j and k are *consecutive* if *abs[j-k mod(n+1)]=1*. All subscript expressions here will be taken modulo n+1. For example: k=j+1 when j<n+1 and k=1 when j=n+1. Also, to clarify the figures, we label the nodes i,j,... instead of a_i, a_k ,... .

case 1: j and k are not consecutive.

examine the following DAG :

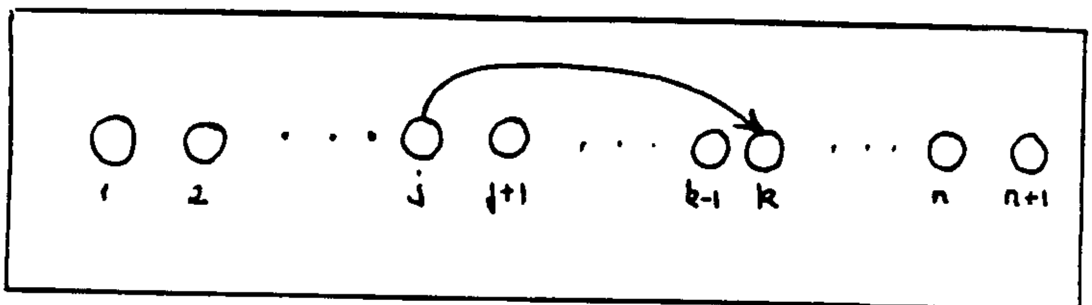


Figure 1

$I(A_i,A_{i-1},A_{i+1})$ holds for all i because i and i+1 are always disconnected.
However, the proposed conclusion $I(A_j,z,A_k)$ is false for any set z (including ∅), because j and k are connected with a direct link.

case 2: j and k are consecutive.

subcase 2.1: Z contains a variable i, i not consecutive to either j or k.

Examine the following DAG:

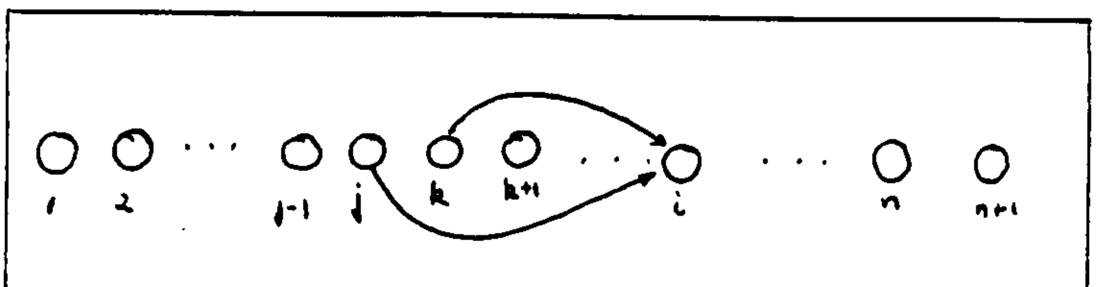


Figure 2

$I(A_j,z,A_k)$ is clearly false. Yet, all independencies in S hold because i is not consecutive to either j or k.

subcase 2.2: Z is a subset of $A_{j-1} \cup A_{k+1}$.

subcase 2.2.1: $Z = A_{j-1} \cup A_{k+1}$
Construct the DAG of Figure 3 where i is an arbitrary node other then j-2,...,k+1,k+2. Such an i always exists for n>7. Once again, S holds, but $I(A_j, A_{j-1} \cup A_{k+1}, A_k)$ is false, because $A_{j-1} \cup A_{k+1}$ activate a path (j, k+1, i, j-1, k) between j and k.

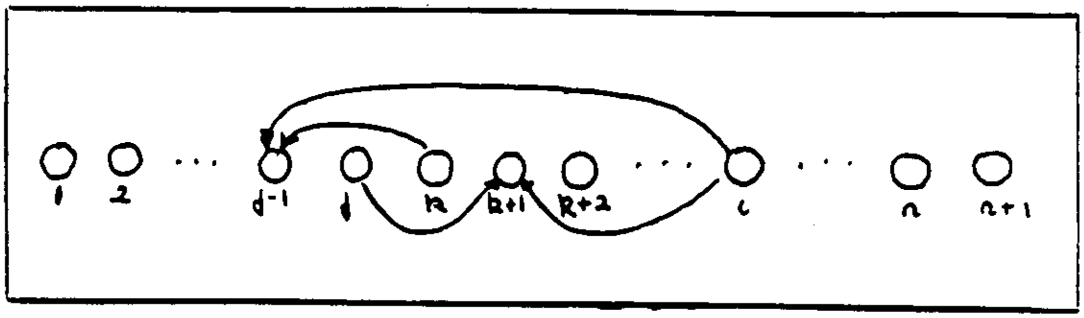


Figure 3

subcase 2.2.2: $Z = A_{k+1}$.
The DAGs of Figures 4 and 5, realize S & $\neg I(A_j, A_{k+1}, A_k)$ for the cases 2<j<n, and j=2 or j=n+1, respectively.

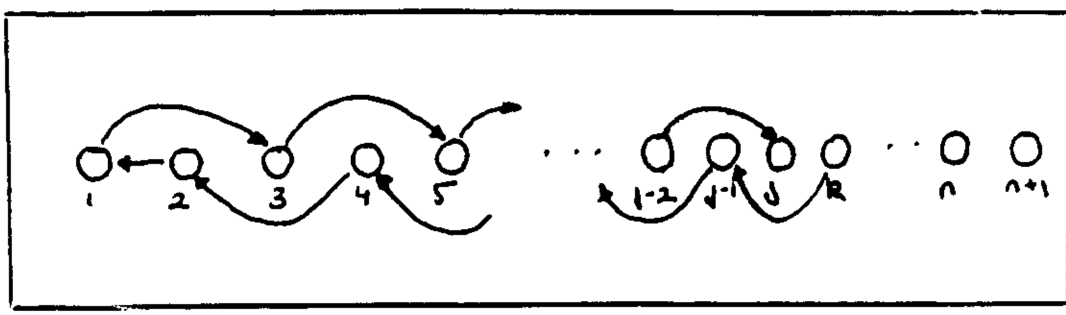


Figure 4

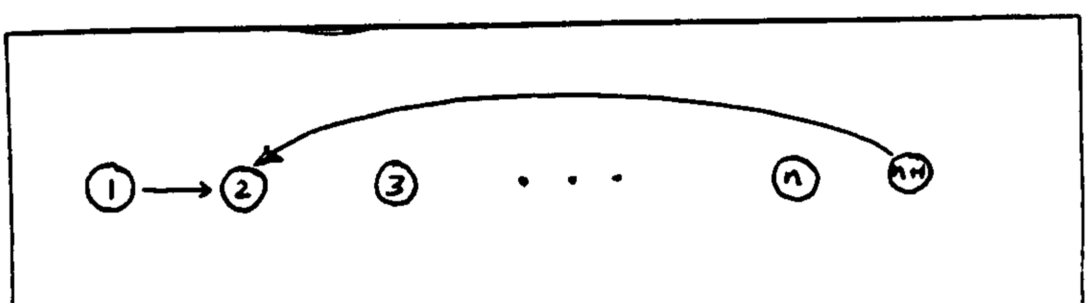


Figure 5

In case j=n, $I(A_j, A_{k+1}, A_k)$ reduces to $I(A_n, A_1, A_{n+1})$ which is a member of **VALID**.

subcase 2.2.3: $Z= A_{j-1}$.

For j=1 the DAG of Figure 5 realizes S & $\neg I(A_1,A_{n+1},A_2)$, while for all other values of j, the statement $I(A_j,A_{j-1},A_k)$ is a trivial consequence because it is a member of S.

subcase 2.2.4: $Z= \emptyset$

For j<n the DAG of figure 4 realizes S & $\neg I(A_j,\emptyset,A_k)$, while j=n and j=n+1 yield $I(A_n,\emptyset,A_{n+1})$ and $I(A_{n+1},\emptyset,A_1)$, which are in **VALID**.

So far, we have shown that all single non trivial consequences of S are listed in **VALID**. To complete the proof, we have to verify that every statement in **VALID** requires all n antecedents of $R_0(n)$, that is, cannot be inferred from any proper subset of S.

Consider the statement $I(A_n,A_1,A_{n+1})$ that belongs to **VALID**. Let S' be a proper subset of S. The following DAG satisfies any S' not containing $I(A_{n+1},A_n,A_1)$ but does not satisfy the consequence $I(A_n,A_1,A_{n+1})$.

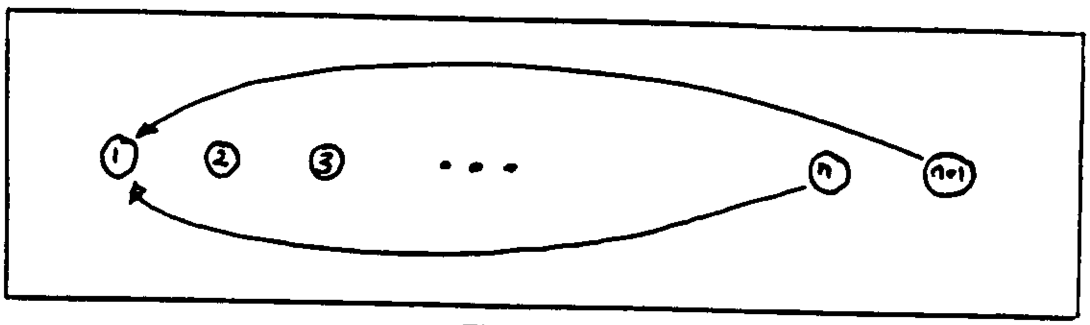


Figure 6

The other possibility is that $I(A_{n+1},A_n,A_1)$ is included in S' but one or more of the first n-1 independencies is not. Define m to be the maximal i such that $I(A_{i+1},A_i,A_{i+2})$ is not in S'. Again, the following DAG satisfies S' but not the consequence $I(A_n,A_1,A_{n+1})$.

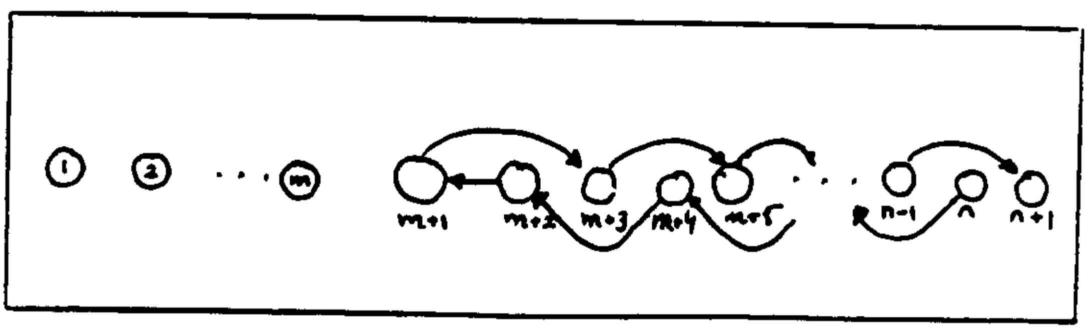


Figure 7

Following similar arguments, the other three statements in **VALID**, can also be shown to require all n antecedents of $R_0(n)$. □

4. EXTENSIONS OF THE INCOMPLETENESS THEOREM.

The non existence of a finite complete set of Horn axioms for DAGs, still leaves us with the question of whether the DAG model is axiomatizable at all. Like the UG model, it might be the case that, one could establish a finite complete set of *disjunctive* axioms for DAGs. Such a set, though, would most likely be computationally intractable and would be useful only as a theoretical tool for studying the properties of the I(x,z,y) relation in the DAG model. We conjecture that even disjunctive axioms would not render the DAG model axiomatizable, i.e complete finite set of disjunctive axioms does not exist.

Conjecture: There is no bounded set of disjunctive axioms for DAGs with the d-separation criteria.

Although, we have not been able to prove this conjecture in general, it can be shown to hold for a large subset of disjunctive axioms.

Before proceeding, we need the following classification of axioms. An axiom

$$I(x_{1,1},x_{1,2},x_{1,3})\ \&\ I(x_{2,1},x_{2,2},x_{2,3})\ \&\ \cdots\ I(x_{n,1},x_{n,2},x_{n,3}) \rightarrow I(y_{1,1},y_{1,2},y_{1,3})\ or\ \cdots\ I(y_{m,1},y_{m,2},y_{m,3})$$

is a *functional-restricted* axiom if every set $y_{i,j}$ is a result of applying boolean functions on the sets $x_{i,j}$. Namely, each $y_{i,j}$ is the result of applying the set-functions: union, intersection and negation on the $x_{i,j}$'s. For example, the weak transitivity axiom:

$$I(x,z,y)\ \&\ I(x,z\cup\gamma,y) \rightarrow I(x,z,\gamma)\ or\ I(\gamma,z,y)$$

is a functional-restricted axiom because x, y and z appear on the left hand side of the implication and γ can be written as $(z\cup\gamma)\cap\neg z$ where both z and $z\cup\gamma$ appear in the left hand side of the implication. On the other hand, the transitivity axiom (which does not hold for DAGs):

$$I(x,z,y) \rightarrow I(x,z,\gamma)\ or\ I(\gamma,z,y)$$

is not a functional-restricted axiom because γ is not functionally depended on x, y and z. Such an axiom is said to be an *unrestricted axiom.*

THEOREM 4.1

There is no bounded complete set of functional-restricted axioms for DAGs with the d-separation criteria.

proof: This proof uses similar techniques to the ones used in the proof of theorem 3.1. Consider the axiom $R_0(n)$ of lemma 3.1. It is sufficient to prove that this axiom can not be reduced to a chain of lower arity functional-restricted axioms. The reason that the proof of lemma 3.2 can not be immediately applied is that there, we only showed that all single consequences need the full set of antecedents S in order to be concluded. However, now we need to show that all disjunctive consequences has this property as well, else a disjunctive consequence might turn $R_0(n)$ to be reducible to disjunctive axioms. Thus, for each disjunction of independency statements π that is not a disjunctive consequence of S, one need, to find a

DAG that obeys S but does not obey the disjunction π. Such a task is infeasible because the length of the disjunction in π is arbitrary and an infinite number of constructions are needed.

To overcome this problem we use, for each term T_i in π, a DAG called D_i that obeys S but does not obey T_i. We then define an operation that collapses the sequence of D_i's to a single DAG denoted $\oplus D_i$ that obeys S but does not obey π. This construction, as will be shown latter, is made possible due to the restriction on the axioms to consist solely of functional-restricted axioms. $\oplus D_i$ is the required DAG showing that π is not a disjunctive consequence of S.

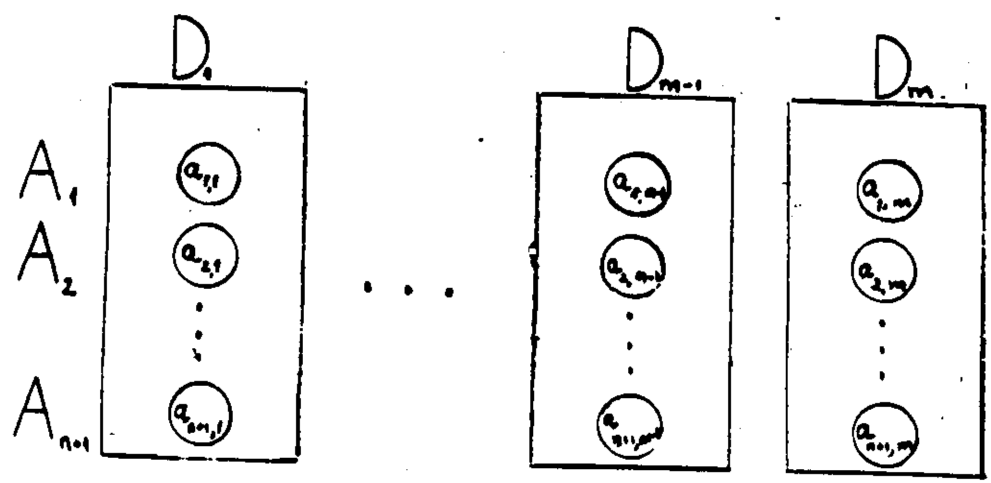
Let A_k be the set $\{a_{k,1}, a_{k,2}, \cdots a_{k,m}\}$ where $a_{i,j}$ are single elements and a let

$$\pi = I(x_1,z_1,y_1) \text{ or } I(x_2,z_2,y_2) \text{ or } \cdots I(x_m,z_m,y_m)$$

be an arbitrary disjunction of statements using the same attributes of S. Assume that all terms in π are not members of **VALID**, otherwise trivialy, π is a disjunctive consequence of S.

The restriction on the axioms to be functional-restricted constrains the disjunctions π that need to be examined. Namely, the sets x_i's, y_i's and z_i's are all functionally depended on the A_k's. However, the A_k's are a partition on the nodes of $\oplus D_i$ and therefore negation and intersection can be reexpressed in terms of union only. Thus w.l.o.g, we can assume that each of the x_i's, y_i's and z_i's is a union of some A_k's. Moreover, due to the decomposition axiom, we can further assume that x_i and y_i are each equal to some A_k.

Consider the following construction of $\oplus D_i$:



$\oplus D_i$ is a collection of m disconnected components D_i, one for each term in π. The nodes of each D_i, denoted N(D_i), are labeled $\{ a_{1,i}, a_{2,i}, \cdots a_{n+1,i} \}$. In the following discussion the term, *I(x,z,y) holds in D_i*, means that $I(x \cap N(D_i), z \cap N(D_i), y \cap N(D_i))$ holds in D_i.

D_i is constructed as in lemma 3.1, in such a way that S holds in D_i and the i-th term of π (name-ly, $I(x_i,z_i,y_i)$) does not hold. For example, if $I(x_i,z_i,y_i) = I(A_1,A_5,A_3)$ then we use the DAG of Figure 1 so that all members of S hold in D_i and $I(a_{1,i},a_{5,i},a_{3,i})$ does not hold. Hence, we have used those ele-ments which are in D_i to construct a path between A_1 and A_3 that is d-separated by $a_{5,i}$. This path is also d-separated by A_5 because all other elements of A_5 reside in components which are disconnected from D_i (This claim is made more rigorous in lemma 4.1).

This construction is made possible because the x_i's, y_i's and z_i's are each a union of some A_k's and therefore have elements in each component D_i. These elements are needed to establish a path, entire-ly within D_i that is not d-separated by z_i.

For example, assume that $I(x_1,z_1,y_1) = I(A_{n+1},\{a_{2,2}\},A_1)$ (Note that z_1 is not functionally depended on the A_k's). For this term our construction is not adequate. The elements of D_1 are not sufficient to realize S and $\neg I(a_{n+1,1},a_{2,2},a_{1,1})$, because $I(a_{n+1,1},\emptyset,a_{1,1})$ (and $I(A_{n+1},\emptyset,A_1)$) is a conse-quence of S. therefore, a link must be drawn between D_1 and D_2. This destroys the disconnectness of the components D_i and therefore, as is shortly shown, no longer can we prove that $\oplus D_i$ obeys S. It should be emphasized, though, that for each disjunction π even, when such terms are present, it is easy to construct the required DAG. This is the reason for our belief that DAGs are indeed non axiomatizable. However, in general, we could not establish the proof without the restriction on the axioms that excluded the need to consider such terms.

We now prove that $\oplus D_i$ satisfies S and not π. For this purpose, we present the following lemma (the proof is given latter).

Lemma 4.1: Let D be a DAG that consists of m disconnected components D_i and let V_i be the nodes of D_i. Then, the following two statements are equivalent.

(a) I(x,z,y) holds in D

(b) For every i, the projection of I(x,z,y) on D_i holds, namely, the statement $I(x \cap V_i\ ,\ z \cap V_i\ ,\ y \cap V_i)$ holds in D_i.

We use this lemma in two ways. First, by our construction, every member of S satisfies (b) there-fore every member of S holds in $\oplus D_i$. Second, each term T_i of π has one component in which T_i does not hold and therefore T_i does not hold in $\oplus D_i$. Hence, π does not hold in $\oplus D_i$.

To complete the proof we show that if π has a term T_{i_0} that is a member of **VALID** then it cannot be inferred from any proper subset S' of S. This is evidently true because for each subset S', a DAG was constructed (Figures 6 & 7) that obeys S' and not T_{i_0}. The use of these DAGs in the same fashion described earlier, shows that π is not a disjunctive consequence of any proper subset of S. Thus, the ax-iom $R_0(n)$ is irreducible to a chain of functional-restricted axioms of lower arity then n. □

Proof of lemma 4.1: Define $x_i \equiv x \cap V_i$, $y_i \equiv y \cap V_i$ and $z_i \equiv z \cap V_i$. Note that since V_i are partitioning the nodes of D, we have $x = \bigcup_i x_i$, $y = \bigcup_i y_i$ and $z = \bigcup_i z_i$.

(a)⇒(b): Assume I(x,z,y). We prove that I(x_i,z_i,y_i) holds for all i. Due to the decomposition axiom $I(x_i,z,y_i)$ also holds. That is, z d-separates all paths between x_i and y_i. It remains to show that also z_i d-separates these two sets. Two cases need to be examined; first, if a path is d-separated because the existence of an element of z on it. Then, this element must also be a member of z_i because D_i is a disconnected component. Second, if a path is d-separated because none of the elements of z reside on it then the removal of some elements of z leaves this path d-separated.

(a)⇐(b): Assume $I(x_i,z_i,y_i)$ holds for all i. Let u,v be two sets each contained in a different disconnected component of D. Then clearly, for each set w the statement I(u,w,v) holds in D. In particular, assign u=x_i, w=z_i and v=z_j j≠i. The statement $I(x_i,z_i,z_j)$ holds for every j, j≠i. Using the composition axiom we obtain $I(x_i,z_i,(\bigcup_{j\neq i} z_j)\cup y_i)$. Applying the weak union property we obtain $I(x_i,\bigcup_i z_i,y_i)$ which reduces to $I(x_i,z,y_i)$. We now use the assignment u=x_i, w=z and v=y_j j≠i to obtain $I(x_i,z,y_j)$ for every j, j≠i. Using composition we obtain $I(x_i,z,y)$. This statement holds for every i, therefore applying the composition axiom again, we obtain that I(x,z,y) holds. □

Note that, in other words, lemma 4.1 states that when a reasoning system based on DAGs is composed of components that are not relevant to each other, then the reasoning can be done separately in each component.

The next lemma shows that the construction of theorem 4.1 is made possible only because we restricted the allowed axioms and thereby restricted the domain of disjunctions of independency statements that need to be examined. In other words, there is no way to construct an operator ⊕ that produces from a sequence of arbitrary DAGs G_i, a DAG for which an independency statement holds iff it holds for each G_i. This lemma is a restricted version of a theorem by Fagin ([9]). The original theorem is stated in a more general terminology that is useful for both relational database theory and Dependency models theory. We supply only the proof of the part that we use, because it is closely related to the construction of of theorem 4.1.

Lemma 4.2: Let S be a set of independency statements. The following properties of S are equivalent.

(a) There is an operation ⊕ that maps indexed families of models into models, such that if σ is a independency statement in S, and if $R_i : i\in I$ are models, then σ holds for $\oplus\langle R_i : i\in I\rangle$ iff σ holds for each R_i.

(b) Whenever Σ and $\sigma_i : i\in I$ are subsets of S, then σ_1 *or* σ_2 *or* $\cdots$ σ_n is a consequence of Σ iff exists an i such that σ_i is a consequence of Σ.

Proof:

(a)⇒(b): By contradiction assume σ_1 *or* σ_2 *or* $\cdots\sigma_m$ is a consequence of Σ and that each σ_i is not a consequence of Σ. Then for each σ_i there exist a model R_i that obeys Σ but does not obey σ_i. Consider the model $\oplus R_i$. This model obeys Σ but does not obey any σ_i, contradicting our assumption that σ_1 *or* σ_2 *or* $\cdots\sigma_i$ is a consequence of Σ. □

The application of this lemma for the DAG model is straightforward. Consider the axiom (2.e) that holds in DAGs. This axiom does not satisfies the conditions in (b). Lemma 4.1 assures us that an operation $\oplus$ that meets the requirements in (a), does not exist for DAGs.

Note that an equivalent statement for (b) is that any set of axioms can be replaced by an equivalent set of Horn type axioms. This observation can be immediately exploited to extend the result of Parker and Parsaye ([5]). They have proved for a dependency model called EMVD that there is no finite complete set of unrestricted Horn axioms. However, since an operation $\oplus$ that satisfies the requirements of lemma 4.1 exists ([7],[9]) it is concluded that there is no finite complete set of unrestricted disjunctive axioms for the EMVD dependency model. This result,though obvious in this context, was never explicitly stated neither in [5] nor in [6].

5. CONCLUSIONS

In this paper we have shown that there exist no finite complete set of Horn axioms for DAGs. We extended this theorem to cover a subclass of disjunctive axioms, which we call functional-restricted axioms. This result can be classified as a "negative" result, since the existence of a complete set of Horn axioms in relational database lead in the past to important results. In [10] (as well as in [5]) such a set is used to solve a problem known as the *membership problem*. This problem, when translated to dependency model terminology, becomes the problem of whether a specific independency statement is a consequence of a given set of independency statements. The solution of the membership problem enabled database researchers to construct a relational database model from a given set of constrains ([8]). This is equivalent to the problem of finding a UG (or DAG) that best satisfies a given set of independency statement.

The similarities between relational database theory and dependency models theory were intensively used in this paper. We have used a similar construction of Parker and Parsaye in establishing theorem 3.1. Lemma 4.2, due to Fagin, was found useful in clarifying the difficulties we encountered trying to prove the incompleteness conjecture. These similarities motivate a search for a more general setting in which results from database theory can be stated side by side with results from dependency models theory. In such a setting results from both fields become readily available for each other. A step towards this direction was made in [9], where Fagin introduced a terminology that is general enough to reflect results from one field to another. In our present research we expand this terminology in order to explore further properties of dependency models.

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