NON-UNIFORM STRUCTURES AND SYNCHRONIZATION PATTERNS IN SHARED-CHANNEL COMMUNICATION NETWORKS

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by

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CHAPTER 1
Introduction

Shared-channel communication systems have been developed and extensively studied during the last decade. In such systems a broadcast communication channel is to be shared among a group of geographically distributed stations each demanding access to the channel in a random fashion. The most common shared-channel communication networks are: 1) *Satellite networks*, in which ground stations communicate with each other by a radio channel and by using a satellite as a repeater. 2) *Packet radio networks*, in which many stations communicate with each other by a ground radio channel. 3) *Local area networks*, in which many stations, geographically close to each other, use a cable communication media* (like a coaxial cable or a fiber optic cable) to communicate with each other.

The main issue in designing shared-channel networks is the design of access schemes. An *access scheme* is a set of rules which determines for each station under which circumstances it is allowed to transmit. The main goal of an access scheme is to efficiently control the stations' transmissions. Two main approaches are commonly used to control access to the channel:

1. Random access schemes.

2. "Organized", conflict free schemes.

In a scheme of the first type any station tries to acquire the channel without completely coordinating its action with its neighbors. Such schemes include ALOHA [Abra70], Slotted-ALOHA [Abra73] and CSMA (see, for example, [Klei76]). While the performance of these schemes is very good for low traffic demand (immediate response) they tend to become poor at times of high demand. This is because collisions are more likely to occur when many stations try to access the channel simultaneously (a *collision* occurs if two or more stations transmit at the same time).

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*The distinction between radio networks to local area networks is not always very sharp. In some cases a local area network may use a radio channel as a communication media; in this case the system may be considered both as a local area network and as a packet radio network.

In a scheme of the second type the system is "organized". Only one station at a time is permitted to transmit, so the access scheme is conflict (collision) free. Such schemes include MSAP [Scho76] (token-ring), TDMA (see, for example, [Klei76]), EXPRESS-NET [Frat81] and others. The performance of these schemes is good for heavy traffic and poor (due to a relatively long response time) for light traffic.

In this work we study a special property of shared- channel communication networks, namely, the *self-synchronizing* property. We take a clue from nature, in which we often observe that a set of many randomly behaving particles may display a self organized global motion. Trying to imitate this phenomenon, we look for these features in the behavior of shared-channel communication networks.

The advantage of this behavior is twofold. First, if the system is self synchronizing, then no external synchronization mechanism is required. Second, such a system should behave efficiently both for heavy and light traffic: when the traffic is light, the station (as equivalent to the independent particle) obeys its own random demands, without conflicting with others. When the traffic is heavy, the behavior of the stations becomes organized so that the number of conflicts is minimized. In a sense, this behavior is a natural mixture of the two basic access approaches: the random access approach at low load and the conflict free approach at heavy load.

Synchronization effects have not been studied extensively in the context of shared-channel communication networks. The complexity of the models (see a detailed discussion of this issue in section 1.1) and the limitations of the available analysis tools have caused researchers to make simplifying assumptions which "hide" (or disregard) these features. The most common of these simplifying assumptions are: 1) Uniformity of the network structure. 2) Independence of events. It is the absence of much previous study that makes this research important. Features that were not studied before (or even were not observed!) may have a significant impact on the performance of communication systems. In general, new access and routing schemes are developed (and implemented) on the basis of understanding system behavior. Therefore, it is important to know how close the existing models (and their analyses) are to the behavior of the real system. Uncovering the synchronization effects in shared-channel communication networks will contribute to a better understanding of the system behavior.

In fact, the results reported in this dissertation show that in many cases the predictions of the simplifying-assumption models are misleading. For example, for some systems the simplifying model predicts very poor performance at high loads; in contrast, this study shows that, due to synchronized behavior, the same system performs almost perfectly at high loads.

We conclude that synchronization properties must be considered when shared-channel communication networks are analyzed. Moreover, it is important to consider these features in the design of shared-channel networks. It is also concluded that simplifying assumptions about uniformity of the structure and about independence of events may cause significantly wrong predictions, and thus should be considered cautiously.

We choose to study the synchronization properties by the means of case studies. These cases are shared-channel communication networks that possess synchronization properties. Since each of these systems is an important topic for itself we study them in full detail, revealing, through the research, the synchronization properties. A description of these case studies is given in section 1.2.

1.1 Motivation, Difficulties and Solution Approaches

In contrast to the assumptions commonly used in the analysis of communication networks, namely independence of events and uniformity, the two properties inherently related to synchronized systems are quite the opposite, namely:

1. Non-uniform structure.

2. Events which are statistically correlated to each other.

In a non-uniform structure some station may be distinguishable from the others. In contrast, in a uniform structure it is hard to distinguish one station from the others; this property is strongly related to the claim that in a symmetric network (no node identifiers) there is no distributed deterministic way to elect a node [Gall79]. To contrast the uniformity property with synchronization note that the elements (stations) in synchronized systems are distinct from each other at a given time (e.g., one transmits, the others are silent). For this reason we conjecture that synchronization features are strongly related to non-uniform structures, and that it is very unlikely to observe these features in uniform structures.

The second property (i.e., correlation) simply states that in a synchronized system the events occurring at time t are correlated to the events occurring at time $\tau < t$.

Not accidently, synchronization patterns have not been deeply studied in communication networks. The reason is that both the non-uniformity property and the correlation property were rarely represented in previous models. The common simplifying assumptions, namely, the independence of events and the structure uniformity did not allow these properties to show up in the models.

It should be clear that any of these "simplifying assumptions" is usually necessary for the simplicity of the analysis in all the models where it is used. The limitation of queueing theory and of other analysis tools do not allow us to easily analyze a system where the events are non-independent (for example the reader can refer to [Yemi80] where a complete chapter deals with the difficulties of solving two dimensional Markov-chains). So, for the sake of "elegant" analysis, simplifying assumptions usually have to be made.

From the above discussion it is implied that our task of revealing synchronizing patterns is not easy. In order to observe these properties we have to abandon some of the "good" assumptions, an action which will make the analysis difficult. For this reason we will not always be able to achieve "elegant" results. Rather, in many of the cases where queueing theory tools will be found to be too weak we may have to resort to some of the following means:

1. Apply some other relaxing assumption that still preserves the properties to be observed.

2. Apply approximation or simulation techniques.

We are aware of the lack of elegance of these techniques but consider them as the "price" that has to be paid in order to better understand communication-network behavior.

Due to the difficulties in the analysis, for some of the systems to be analyzed it was required to solve some basic problems in queueing theory. For this reason this dissertation consists of two logical parts: The first part is a theoretical part and is devoted to the study of several basic queueing systems. The second part deals with studying the synchronization feature in several shared-channel communication networks. The queueing theory results derived in the first part are used in the second part where we analyze shared-channel communication networks.

1.2 The Case Studies

Three shared-channel communication systems are considered in this work. The first system is a one-hop* network where the access scheme used is exhaustive slotted ALOHA. According to this scheme, a station which successfully transmits a unit of information (called *a packet)* will have the exclusive right to continue transmitting other packets without being interrupted by the other stations. Only after this station *exhausts* its buffer, are the other stations allowed to access the channel. This system can be thought of as "control synchronized" (in contrast to naturally synchronized) since the exhaustive scheme implies that the stations take turns in their transmissions so that the system is actually synchronized.

*A *one-hop network is a network* where every station hears all the other stations. In contrast, a *multi-hop* network is a system where a given station hears only some of the other stations, which are called the *neighbors* of this station.

The second system is a directional tandem in a multi-hop radio environment where the access scheme used is slotted ALOHA. The tandem is a basic structural component of any multi-hop radio network. Thus understanding the tandem behavior is very important for the analysis of multi-hop radio networks. This study reveals that at high loads the tandem network tends to synchronize itself and thus to perform very efficiently [Yemi80]. This is in contrast to the common belief that the performance of shared-channel networks significantly degrades at high loads. The synchronization feature analyzed in this study may lead to the invention of efficient access and routing schemes tailored for the multi-hop radio network.

The third and the last system is a very-fast-bus network. In this system all stations are connected to a very fast bidirectional bus through which they communicate with each other. The common belief about bus networks is that the performance of these systems under random access schemes (like CSMA or CSMA/CD) degrades when the speed of transmission increases. According to this belief the performance of the CSMA access scheme will be very poor when implemented on a very fast bus system. In contrast to this belief, our study reveals that in a very-fast bus system with a high load, the system tends to synchronize itself. As a matter of fact we discover that the system throughput monotonically increases with the offered load and reaches a value of 1 at very high load.

1.3 Structure of this Work and Summary of Results

As mentioned above this dissertation consists of two main parts: The first part, containing chapters 2, 3 and 4, deals with solving general problems in queueing theory. The second part, containing chapters 5, 6 and 7, deals with the case studies discussed above. The solutions derived in chapters 2, 3 and 4, are used in the analysis carried in chapter 5. Due to the nature of this work the previous work related to each topic is separately reviewed in each of the chapters.

1.3.1 Delay Analysis of a Queue with an Independent Starter (Chapter 2)

A single-server queueing system in which a start-up delay is incurred whenever an idle period ends and a single-server queueing system in which the server takes vacation periods when the system is idle are both analyzed in chapter 2. In all systems analyzed in this chapter the start-up delays (or the vacation periods) are assumed to be random variables *independent* of the system state.

The main result of this chapter states that the delay distribution in the queue with starter is composed of the direct sum of two independent random variables: 1) The delay in the equivalent queue without starter, called the *original delay*. 2) The *additional delay* suffered due to the start-up delay. Using this *decomposition* property, it is easy to derive the distribution of the delay suffered in the system with starter. This analysis is done for systems (both discrete and continuous) where the interarrival times possess the memoryless property*. In the second part of this chapter, using the decomposition approach, we analyze the M/G/1 system with vacation periods. It is first shown that M/G/1 with vacations is just a special case of M/G/1 with starter, so that the delay in M/G/1 with vacations can be easily found by using the formula for the delay of M/G/1 with starter. Second, using geometric arguments it is explained why the additional delay in the vacation system is distributed as the residual life of the vacation period.

1.3.2 Delay Analysis of a Queue with a Non Independent Starter (Chapter 3)

Chapter 3 is a natural extension of the study done in chapter 2. Again, in this chapter, we analyze the delay of a single server queue with a starter. However, in contrast to the systems studied in chapter 2, here the start-up delay is *not assumed to be independent of the system state.* Rather, it is assumed that the length of the start-up duration *depends on the arrival process.*

The analysis method used in this chapter is similar to the one used in chapter 2. Using the *decomposition property* presented in chapter 2, it is relatively simple to analyze the delay in a single server queue with a non independent starter.

Two types of systems are analyzed: 1) A system where the start-up delay depends on the amount of work (or the number of customers) arriving to the system at the beginning of the start-up period. 2) A system where the start-up delay depends on the length of the idle period preceding the start-up operation. As in chapter 2 the analysis is done for systems (both discrete and continuous) where the interarrival periods possess the memoryless property.

For these systems we derive the expected value of the *additional delay* suffered in the system. This value can be added to to the expected value of the *original delay* (which, for most systems without a starter is easy to derive) to yield the expected value of the *total delay* in the system with starter. In addition, in the cases where the original delay cannot be derived, the expression for the expected value of the additional delay has its own importance if different starters are compared to each other. It is shown that these results can be easily applied to systems with bulk arrivals as well as for systems with single arrivals.

A deeper study is done on the M/G/1 system with a starter where the start-up delay depends on the service time of the *first customer* (a customer arriving at an empty system). For this system we derive the Laplace-Stieltjes transform (LST) of the *total delay* suffered in the system.

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*The delay analysis of an M/G/1 system with an independent starter has been done previously using other analysis methods. Using the decomposition method we rederive this result.

The results of this analysis and the results reported in chapter 2 are applied in chapter 5 to the analysis of some exhaustive slotted ALOHA systems.

1.3.3 The Analysis of Random Polling Systems (Chapter 4)

In chapter 4 we analyze the behavior of *random polling systems*. The polling systems considered consist of N stations, each of them equipped with an infinite buffer, and of a single server who serves them in some order. In contrast to previously studied polling systems where the order of service used by the server is *periodic* (and usually *cyclic*), in the systems considered here the next station to be served after station i is determined by *probabilistic means*. More specifically, according to the model considered in this chapter, after serving station i the server will poll station j ($j=1,2,\cdots,N$) with probability p_j.

The model considered here is a discrete time model where the service time of a customer is assumed to be fixed and equal to the discrete time unit. The arrivals to each station are assumed to be bulk arrivals with bulk size taken from an arbitrary distribution, and the size of the bulk arriving at time t_1 independent of the size of the bulk arriving at time t_2 (for $t_1 \neq t_2$).

Three service policies are considered in this chapter: 1) *Exhaustive policy.* 2) *Gated policy.* 3) *Non exhaustive policy.* For all these service policies we derive a closed form expression for the expected delay in the system when the stations are assumed to be symmetric. For non symmetric systems we present a set of N^2 linear equations that must be solved numerically to yield the expected value of the delay. Also derived in this chapter are relations for the z-transforms of the cycle time and of the number of customers found in the system at polling instants.

The results reported in this chapter are applied in chapter 5 in the analysis of exhaustive ALOHA schemes.

1.3.4 An Analysis of the Exhaustive Slotted ALOHA System (Chapter 5)

In this chapter we study the queueing behavior of exhaustive slotted ALOHA, a method which is used to control the transmission of N stations in a one-hop environment. Our main goal in this chapter is to derive the expected delay observed in this system. This is our first case study.

For a two station system where the stations know that $N=2$ we derive a closed form expression for the expected value of the system delay. This analysis is done partially by using the queue with starter approach and the results derived in chapters 2 and 3, and partially by using a Markov Chain approach.

For an N station system we derive a closed form expression approximating the expected delay in a symmetric system under heavy load conditions. To derive this expression we emulate the system by a random polling system and use the results derived in chapter 4.

For a two station system where the stations do not know that $N=2$ we use the two results reported above: 1) The expression derived for the two station system (where the stations know that $N=2$) is used as a low load approximation and as a lower bound. 2) The expression derived for the N station system is used as a heavy load approximation. To approximate the expected delay in the middle range of the load we use a linear combination of the two approximations. The values predicted by this approximation are compared to simulation results and found to be very accurate.

1.3.5 Synchronisation properties in the Behavior of a Slotted ALOHA Tandem (Chapter 6)

In chapter 6 we analyze the throughput of a slotted ALOHA directional tandem in a multi-hop packet radio environment. This second case study considers a model consisting of N stations each of which is equipped with an infinite buffer and each of which transmits to its downstream neighbor. The access scheme used is symmetric slotted ALOHA according to which a station whose buffer is not empty will transmit with probability p. Our point of departure is a previously reported result [Yemi80] which states that when $p=1$ the tandem is fully synchronized and the system throughput is 1/3, which is the maximum achievable throughput in a tandem. We extend this result and show that for $p<1$ the system is "partially synchronized" and derive an expression approximating the throughput (S) for high values of p. This expression is:

$$S \cong \frac{1}{3\cdot\left(2-p^{N/3}\right)}$$

For low values of p we approximate the throughput by the following expression:

$$S \cong p(1-p)^2$$

For the medium range of p we suggest an approximation to the system throughput by using a linear combination of the two expressions given above. The approximations suggested are compared to simulation results and are found to be very accurate for most practical tandems (tandems which consist of less than 20 stations).

The results reported in this chapter show that when we account for synchronization the tandem throughput monotonically increases with the transmission probability (p). This is in contrast to what could be implied from previous studies where the stations are assumed to behave independently of each other (i.e., not accounting for synchronization).

1.3.6 On the Behavior of a Very Fast Bidirectional Bus Network (Chapter 7)

In this chapter we study the behavior of the very fast bidirectional bus system (out third case study). We assume that the system consists of N stations located on a bidirectional bus and analyze the system throughput for different transmission policies. A major assumption in this analysis is that the bus is very fast, so that the time for a packet (one unit of transmitted information) to propagate from one station to its neighbor is equal to or greater than the transmission time of the packet. This study is challenged by the results reported in previous studies which predict that certain access schemes like CSMA (which uses the carrier sense mechanism) will perform very poorly in this environment. In contrast to this prediction we show that due to synchronization properties observed in this system, the system throughput is relatively high.

The first part of this chapter deals with the theoretical limitations of the system. In this analysis we solve for the system capacity (defined to be the maximum achievable throughput under a *carefully selected transmission schedule*). A surprising result of this analysis states that when the stations are not forced to obey the carrier sense rule the system capacity is approximately 2. More precisely, we show that the capacity (denoted by C) is bounded as follows:

$$2-\frac{2}{N} \leq C \leq \min(2,\frac{N}{2})$$

where N is the number of stations. On the other hand for a system where the carrier sense mechanism is enforced we show that the system capacity is exactly 1.

The second part of this chapter, deals with the system throughput under *stochastic arrivals*. For a system where carrier sense mechanism is not enforced it is shown that the system throughput is identical to the throughput of the slotted ALOHA scheme when applied in a satellite network (or in a one-hop radio network). According to this result the total throughput (denoted by S) in a fully symmetric system is given by:

$$S = NG(1-G)^{N-1}$$

where N is the number of stations and G is the offered load of each station. For a system where the carrier sense mechanism is enforced we cannot derive a closed form expression for the throughput. Rather, we introduce an approximation yielding a set of N-1 non-linear equations which must be solved to give the system throughput. The results predicted by the approximation method are compared to simulation results and found to be accurate. The behavior observed for this system is rather surprising: we see that the total system throughput monotonically increases with the offered load, for every value of the offered load. This results from an interesting synchronized behavior of the system, a behavior according to which the system becomes more and more synchronized as the offered load increases. In contrast to what could be implied from previous studies these results provide evidence that the very fast bus system is very stable and does not need any artificial mechanisms to control its stability.

CHAPTER 2

A Queue with an Independent Starter: Delay Analysis

A queueing system in which a start-up delay is incurred whenever an idle period ends and a queueing system in which the server takes vacation periods are both analyzed in this chapter. In all systems analyzed in this chapter the start-up delays (or the vacation periods) are assumed to be random variables *independent* of the system behavior. It is shown that the delay distribution in the queue with starter is composed of the direct sum of two independent variables: 1) The delay in the equivalent queue without starter. 2) The additional delay suffered due to the starter presence. Using this decomposition property, it is easy to derive the distribution of the delay suffered in the system with starter. This analysis is done for systems (both discrete and continuous systems) where the interarrival times possess the memoryless property. Using this approach, we then analyze the M/G/1 system with vacation periods. It is first shown that the M/G/1 with vacations is just a special case of the M/G/1 with starter, so that the delay in the M/G/1 with vacations can be easily found by using the formula for the delay of the M/G/1 with starter. Second, using geometric arguments it is explained why the additional delay in the vacation system is distributed as the residual life of the vacation period.

2.1 Model and Previous Work

In the following we consider a queueing system with a "starter." In such a system the server is "turned off" whenever it becomes idle. When a customer arrives at an idle system, it cannot be served immediately; rather, an additional (random) amount of time is required to start the "cold" system before the new "first" customer can be served. The length of the period required to start the system up, is assumed, in this chapter, to be a random variable independent of the system behavior. Customers who arrive at a "hot" system (i.e. one with at least one customer either in service or in the queue) will join the queue and be served in turn as in a simple queueing system.

The model for a queueing system with special consideration of a case where the server becomes idle is not new. Miller [Mill64] analyzed the case where the server goes on a vacation ("rest period") of random length whenever it becomes idle. He also considered a system where the server behaves normally but the first customer arriving at an empty system has special service time. Scholl [Scho76], and Scholl and Kleinrock [Scho83] analyzed the "server with rest periods," using another approach. As a special case for rest periods, Scholl [Scho76] considered a queueing system with a starter (or, in his words, "a system with initial set-up time"). In both papers, the analysis is done on M/G/1 queues. These types of systems were reported also by

Cooper [Coop70], Heyman [Heym77], Levy and Yechiali [Levy75], Shanthikumar [Shan80], Avi-Itzhak Maxwell and Miller [Avi-65] and Van Der Duyn Schouten [Scho78].

The need for studying *a queue with starter* for slotted (i.e. discrete time) systems, and the fact that previous studies analyzed only M/G/1 (i.e. continuous time) systems, motivated us to study the queue with starter again. The emphasis in this chapter, is on developing a novel approach to study this system. This approach will compare the delay suffered by a customer in a usual queueing system versus the delay in a system with starter. Instead of deriving the delay in the queue with starter directly, we find the *additional delay* suffered due to the presence of the starter. Moreover, we show that the *additional* delay in the system with starter, is independent of the delay in the system without starter. Using the independence property, it is then easy to calculate the total delay in the system with starter: *this is simply the direct sum of the delay in the queue without starter and the additional delay calculated above.*

The approach described above, is found to be very powerful in analyzing systems similar to the queue with starter. In chapter 3, using the same approach, we analyze a queueing system with starter where the length of the start-up period *depends* on the arrival process (unlike the system analyzed here where the start-up time is independent of the arrival process). In chapter 4, the results reported in this chapter (and in chapter 3) are used to derive the delay in an exhaustive ALOHA system. The fact that the delay in the queue with starter can be calculated as the (independent) sum of two independent random variables, one of them representing the delay in the queue without starter and the other representing the additional delay, makes the analysis in those chapters relatively simple.

As stated above, in contrast to previous studies which analyzed M/G/1 systems, the emphasis here is on studying slotted systems. In section 2.3 we analyze the delay in a slotted queue with starter. In this analysis we derive the z-transform of the delay in this system. For the sake of completeness, we use our approach to rederive the delay in an M/G/1 queue with starter and find agreement with Scholl's results. In sections 2.4 and 2.5 we study a system with vacation periods. First, we show that a system with vacation periods is just a special case of the queue with starter. Thus, the delay in this system can be easily found from the delay of the queue with starter (derived above). We then show that the delay of an M/G/1 with vacation periods *is exactly* of the sum of two independent random variables:

- the delay in an M/G/1 without vacation periods;
- additional delay distributed as the residual life of the vacation period.

Lastly, it should be mentioned that some of this work (reported first in [Levy83]) has been reported, in parallel, in two independently written papers. First, Fuhrmann [Fuhr83] showed that the delay in the queue with vacation periods consists of the sum of two independent random variables:

- the delay in an M/G/1 without vacation periods;
- additional delay distributed as the residual life of the vacation period.

This result is identical to what we show in section 2.5. Nevertheless, the method used in [Fuhr83] to prove this property is rather different from the method used in our analysis. Second, Doshi [Dosh83] addressed the decomposition property in both the queue with starter and the queue with vacations. The model used in that report is a continuous time model of a GI/G/1 queue. The emphasis in [Dosh83] is on studying the queue with vacation periods, while the queue with starter is considered as a special case of the queue with vacation periods.

2.2 Notation, Definitions and System Description

In the following we analyze our queueing system by means of the unfinished work in the system. We define:

$U(t) \triangleq$ unfinished work in the system at time t;

$\triangleq$ remaining time required to empty the system of all customers present at time t.

We use the usual notation:

$C_n \triangleq$ the nth customer

$\tau_n \triangleq$ arrival time of C_n.

$t_n \triangleq \tau_n - \tau_{n-1} =$ interarrival time between C_{n-1} and C_n.

$c_n \triangleq$ service time of C_n.

In figure 2.1a we plot the behavior of $U(t)$ versus t in a simple queueing system. This system will be called *system-A*. As described in, [Klei75] $U(t)$ can be viewed as the virtual waiting time, i.e., if the service policy is first-come-first-served, the waiting time of customer C_i is $U(\tau_i)$ (all the work residing in queue when C_i arrives). We also use the terms "busy period" and "idle period" to represent durations in which the server is busy or idle (respectively). The busy period durations are denoted by $Y_1, Y_2, Y_3,...$ and the idle period durations are denoted by $X_1, X_2, X_3,...$

We can now switch to the queue with starter system and call it *system-B*. In figure 2.1b we plot $U(t)$ versus t in system-B. In order to compare the systems, we use the same arrival behavior in both systems. This means that the sets of arrival instances ($\{\tau_i\}$) and service times ($\{c_i\}$) are identical in both systems. In this figure the dashed line represents system-A and the solid line system-B. The difference (denoted by D) represents the *additional delay* suffered in system-B.

In figure 2.1b we note that customer C_1 arrives to an empty system and thus suffers an additional delay D_1 due to a *cold start*. Note that C_2 and C_3 suffer exactly the same additional delay. When C_4 arrives, it finds the system idle, and suffers the additional delay of a second cold start (D_2), which is not necessarily identical in length to (D_1). However, another behavior is

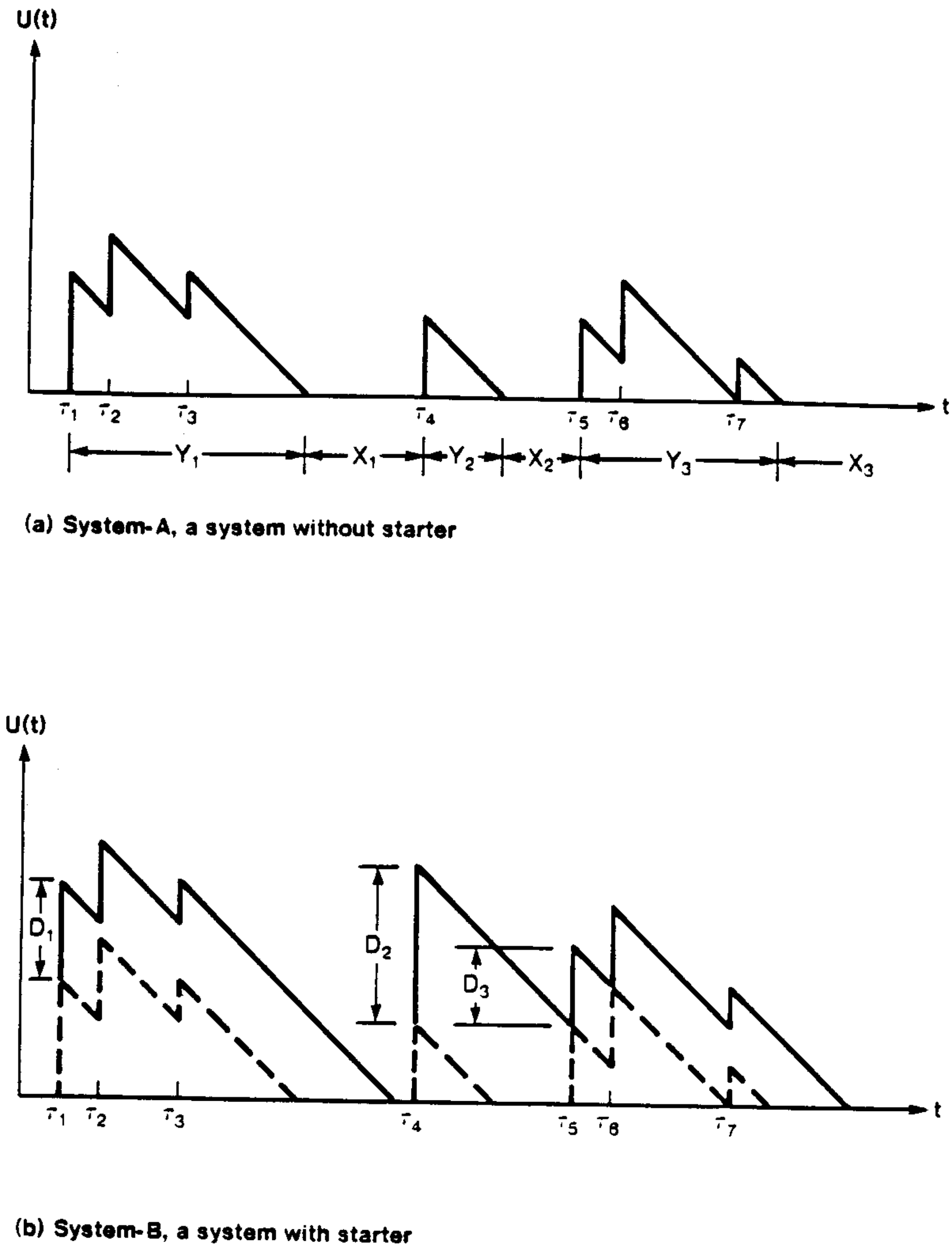


Figure 2.1: The unfinished work in the system (with and without a starter)

observed when C_5 arrives. Since $D_2 > X_2$, C_5 finds the system busy, and a cold start is not required. Nevertheless, C_5 is still subjected to an additional delay, which is $D_2 - X_2$. Again, we note that C_6 and C_7 suffer the same additional delay as C_5.

Keeping this in mind, we now turn to the analysis of system-B.

2.3 The Analysis of System-B, a Queue with Starter

As we mentioned before, this analysis will be done by comparison to system-A. Thus, we shall compare the behavior of systems A and B under the same arrival pattern.

In addition to the notation used above, the following are also used:

X_i = length of the ith idle period (under system-A).

S_i = length of a cold start (if any) corresponding to the ith busy period .

D_i = actual additional delay suffered by the first customer of the ith busy period.

The reader should note that, even though we deal with system-B, we still consider busy periods according to their appearance in system-A. Thus, all the above notation relates to busy periods as viewed in system-A, e.g., the ith busy period is the ith busy period in system-A.

We start our analysis by observing the basic properties of the queue-with-starter system.

2.3.1 The Basic Properties of the System

The assumptions required for the general analysis are the following:

1. The length of an interarrival time, t_i, is independent of the length of any other interarrival time, t_j $(i \neq j)$. The service time of an arbitrary customer, c_i, is independent of the service time of any other customer, c_j $(i \neq j)$. Service times are independent of interarrival times, so c_i is independent of t_j for all i and j.

2. The length of a cold start, S_i, is independent of the length of any other cold start S_j (for any $i \neq j$).

3. The length of a cold start is independent of of the series $\{t_n\}$ and the series $\{c_n\}$. This implies that S_i is independent of the series $\{X_j\}$ and $\{Y_j\}$.

The first assumption is very common for most queueing systems. The second and third assumptions simply state that the length of a cold start is chosen independently of system-A and of the length of other cold starts.

Next, we show how to calculate the additional delay suffered by the *first customer* of busy period i. The additional delay suffered by this *first customer* can recursively be calculated from the following equation:

$$D_1 = S_1 \tag{2.1a}$$

$$D_{i+1} = \begin{cases} D_i - X_i & \text{if} \quad D_i \geq X_i \\ S_{i+1} & \text{if} \quad D_i < X_i \end{cases} \tag{2.2b}$$

The first line in the recursion (2.2b) represents the case where the first customer of busy period i (from system-A) finds system-B busy, while the second line represents the case where this customer finds the system idle, and its additional delay is due to an independent cold start. The basis of the recursion, D_1, is clearly the first cold start of the system. In the following sections, we will use this recursion to calculate the limiting distribution of D_i.

While the additional delay suffered by a "first customer" is an important measure, our main interest is the additional delay suffered by an arbitrary customer. In the following we show that the distributions of these two measures are *identical.*

THEOREM 2.1: If customers C_i and C_j belong to the same busy period in system-A, they suffer exactly the same additional delay in system-B.

The proof is omitted here for its simplicity; examination of figure 2.1 can convince the reader.

THEOREM 2.2: D_i is independent of X_i for every i.

Proof: It is clear that D_i is a function only of $X_1, X_2, \cdots X_{i-1}$ and of $S_1, S_2, \cdots S_{i-1}$. Since X_i is independent of all these variables it is also independent of D_i. ∎

The following theorem states that the additional delay a customer suffers in the system with starter is actually independent of the delay it suffers in the system without starter.

THEOREM 2.3: The additional delay suffered by an arbitrary customer in system-B is statistically independent of the delay this customer would suffer in the equivalent system-A.

Proof: Consider an arbitrary customer, C_i. Let j be the busy period in which C_i is served in system-A and let C_k ($k \leq i$) be the first customer served in this busy period. From theorem 2.1 the additional delay suffered in system-B by C_i and C_k is the same. Thus, we have to show that the additional delay suffered by C_k in system-B is independent of the delay C_i suffers in system-A. It is clear that the delay suffered by C_i is a function only of the interarrival times and the service times that "belong" to busy period j. Namely, the series $t_{k+1}, t_{k+2}, \cdots t_i$ and the series $c_k, c_{k+1}, \cdots c_i$. On the other hand, the additional delay suffered by C_k is only a function of the system

behavior prior to τ_i (the starting time of busy period j). Specifically, this is a function only of the sequence $t_2,t_3, \cdots t_k$, the sequence $c_1,c_2, \cdots c_{k-1}$ and the sequence $S_1,S_2, \cdots S_j$. Now, since the group of variables on which the delay (in system-A) depends and the group of variables on which the additional delay depends are mutually exclusive, and due to assumptions 1 and 3, these groups are statistically independent of each other. Thus, the additional delay suffered in system-B is independent of the delay suffered in system-A. ■

This theorem is very powerful and is a key result of our analysis. It now allows us to study the total delay suffered in the system with starter in three steps: 1) Derive the delay suffered in the system without starter. 2) Derive the additional delay suffered in the system with starter. 3) Convolve the distributions of the two delays to yield the total delay in the system with starter.

The next theorem states that the additional delay suffered in system-B by the customers of a given busy period (according to system-A) is independent of the number of customers served in this busy period.

THEOREM 2.4: D_i is independent of the number of customers served in busy period i.

The proof is omitted for its similarity to the proof of theorem 2.3.

The following corollary is a direct result of theorems 2.1 and 2.4 and states that the limiting distribution of the additional delay suffered by an arbitrary customer is identical to the limiting distribution of the additional delay suffered by a "first customer".

COROLLARY 2.5: The limiting distribution of the additional delay suffered by an *arbitrary* customer in system-B is identical to the limiting distribution of D_i.

2.3.2 Discrete System with General Memoryless Arrivals

The model assumed here is a discrete time model in which time is indexed by fixed length slots. The arrival process can be described as a renewal process, which means that the number of arrivals in slot i is independent of the number of arrivals in slot j for any $i \neq j$. The number of arrivals in a given slot is taken from a general distribution, and the service time is general.

In this section we are interested in the limiting behavior of D_i as i approaches infinity. We recall that time is measured in units of the fixed slot length and define:

$$d_i^j \triangleq Pr[D_j=i] \ , \quad D_j(z) \triangleq \sum_{i=0}^{\infty} d_i^j z^i \ , \quad d_i \triangleq \lim_{j\to\infty} d_i^j \ , \quad D(z) \triangleq \sum_{i=0}^{\infty} d_i z^i \ , \quad \overline{D} \triangleq \sum_{i=0}^{\infty} i \cdot d_i$$

$$x_i^j \triangleq Pr[X_j=i] \ , \quad X_j(z) \triangleq \sum_{i=0}^{\infty} x_i^j z^i \ , \quad x_i \triangleq \lim_{j\to\infty} x_i^j \ , \quad X(z) \triangleq \sum_{i=0}^{\infty} x_i z^i \ , \quad \overline{X} \triangleq \sum_{i=0}^{\infty} i \cdot x_i$$

$$s_i^j \triangleq Pr[S_j=i] \ , \quad S_j(z) \triangleq \sum_{i=0}^{\infty} s_i^j z^i \ , \quad s_i \triangleq \lim_{j\to\infty} s_i^j \ , \quad S(z) \triangleq \sum_{i=0}^{\infty} s_i z^i \ , \quad \overline{S} \triangleq \sum_{i=0}^{\infty} i \cdot s_i$$

$$a_i \triangleq Pr[i \text{ arrivals in a given slot}]$$

In addition, $S^{(1)}(z)$ and $S^{(2)}(z)$, will respectively denote the first and second derivatives of S(z), and $D^{(1)}(z)$ will denote the first derivative of D(z).

Under the above assumptions, it is clear that the random variables X_j, representing the lengths of the idle periods, are independent and identically distributed. Thus, the limiting distribution of X_j is identical to the distribution of X_j. Since the number of arrivals in any slot is independent from slot to slot, then X_j is geometrically distributed (shifted by a slot) with parameter a_0 (the probability of no arrival). For the sake of simplicity, let us use $x = a_0$; so X_j is distributed as follows:

$$x_i = x_i^j = Pr[X_j = i] = (1-x)\cdot x^{i-1} \qquad i=1,2,3... \tag{2.2}$$

From similar arguments, it is clear that the limiting distribution of the length of a cold-start is identical to the distribution itself, so $s_i = s_i^j$.

In the following we solve for D(z). From (2.1) we get:

$$d_i^{j+1} = Pr[D_j - X_j = i] + Pr[D_j < X_j]\cdot Pr[S_{j+1}=i \mid D_j < X_j] \qquad i=0,1,2,... \tag{2.3}$$

Using the independence property between D_j and X_j (theorem 2.2) and the independence between S_{j+1} to D_j and X_j, and using the fact that $x_i^j = x_i$ and $s_i^j = s_i$, we compute d_i^{j+1}:

$$d_i^{j+1} = \sum_{k=1}^{\infty} x_k d_{k+i}^j + s_i \cdot \sum_{k=0}^{\infty} d_k^j \sum_{l=k+1}^{\infty} x_l \qquad i=0,1,2,... \tag{2.4}$$

From (2.4), compute the z-transform of D:

$$D_{j+1}(z) = \sum_{i=0}^{\infty} d_i^{j+1} z^i = \sum_{i=0}^{\infty} z^i \cdot \left[\sum_{k=1}^{\infty} x_k d_{k+i}^j + s_i \sum_{k=0}^{\infty} d_k^j \sum_{l=k+1}^{\infty} x_l \right] \tag{2.5}$$

Substituting (2.2) into (2.5):

$$D_{j+1}(z) = \sum_{i=0}^{\infty} z^i \cdot \sum_{k=1}^{\infty} (1-x)x^{k-1} d^j_{k+i} + \sum_{i=0}^{\infty} z^i s_i \cdot \sum_{k=0}^{\infty} d^j_k \cdot \sum_{l=k+1}^{\infty} (1-x)x^{l-1}$$

Manipulating this and using the definitions for S(z),D(z) gives us:

$$D_{j+1}(z) = \frac{(1-x)\cdot[D_j(x)-D_j(z)]}{x-z} + S(z)\cdot D_j(x) \tag{2.6}$$

Computing D(z), by taking limits, gives:

$$D(z) = D(x)\cdot\left[\frac{1-x+S(z)(x-z)}{1-z}\right] \tag{2.7}$$

Solving (2.7) at z=1, using D(1)=1, S(1)=1 and L'Hospital's rule, we get:

$$1 = D(x)\cdot\left[\frac{S^{(1)}(1)\cdot(x-1)-1}{-1}\right] \tag{2.8}$$

Using $S^{(1)}(1) = \bar{S}$, we get:

$$D(x) = \frac{1}{1+(1-x)\bar{S}} \tag{2.9}$$

Substituting (2.9) into (2.7) gives us the important result:

$$D(z) = \frac{1}{1+(1-x)\bar{S}}\cdot\left[\frac{1-x+S(z)(x-z)}{1-z}\right] \tag{2.10}$$

Expression (2.10) relates the z-transform of the additional delay to the probability of no arrival (x), the z-transform of a cold start ($S(z)$) and the expected length of a cold start ($\bar{S}$). To calculate the z-transform of the *actual* delay suffered in the queue-with-starter, one has to calculate the z-transform of the delay in the equivalent queue-without-starter, and to multiply it by D(z) (as taken from (2.10)). This is true since the additional delay in the queue-with-starter is *independent* of the delay in the queue-without-starter (see theorem 2.3).

Given equation (2.10), it is now easy to compute the expected additional delay. Using the relationship

$$\bar{D} = \frac{\partial D(z)}{\partial z}\bigg|_{z=1}$$

we get:

$$D^{(1)}(z) = \frac{1}{1+(1-x)}\bar{S}\cdot\left[\frac{[S^{(1)}(z)(x-z)-S(z)]\cdot[1-z] + [1-x+S(z)(x-z)]}{(1-z)^2}\right] \tag{2.11}$$

Evaluating (2.11) at $z=1$, using L'Hospital's rule, and further manipulation gives:

$$\overline{D} = D^{(1)}(1) = \frac{1}{1+(1-x)\overline{S}} \cdot \left[\frac{2\overline{S}-S^{(2)}(1)(x-1)}{2}\right] \tag{2.12}$$

Recalling that $S^{(2)}(1)=\overline{S^2}-\overline{S}$, we get:

$$\overline{D} = \frac{2\overline{S}+(\overline{S^2}-\overline{S})(1-x)}{2+2\overline{S}(1-x)} \tag{2.13}$$

We note that the mean of the additional delay depends on the first and the second moments of the cold start and on the probability of at least one arrival (1-x) in a slot.

From corollary 2.4 it is clear that (2.10) and (2.13) represent the additional delay and its expected value for an *arbitrary* customer in the system.

2.3.3 The Behavior of the Mean Additional Delay in the Discrete System

The purpose here is to examine the behavior of expression (2.13) for the expected additional delay suffered due to the existence of the start-up delays.

The behavior of (2.13) when arrivals are rare ($1-x$ approaches 0) is $\overline{D} \approx \overline{S}$. In this situation the distance (in terms of time) between consecutive busy periods is very large, such that almost every busy period suffers a cold start. Therefore, almost all customers of a busy period (usually exactly one) will suffer a "cold start", so $\overline{D} \approx \overline{S}$ and $D(z) \approx S(z)$.

When arrivals are common ($1-x \approx 1$), the length of idle periods is usually 1, and the average of the additional delay is:

$$\overline{D} = \frac{\overline{S} + \overline{S^2}}{2\cdot(1 + \overline{S})}$$

This result agrees with the following simple calculation: Suppose that at busy period i a cold start occurred and that the length of this cold start is j. The additional delay suffered by busy periods $i,i+1,\cdots,i+j$ is $j,j-1,\cdots 0$, respectively (since the length of each idle period is 1); busy period $i+j+1$ will suffer a new independent cold start. Let $\overline{D}(|j)$ be the mean of the additional delay suffered by busy periods which are under the influence of a cold start, the length of which is j. Clearly, $\overline{D}(|j) = \frac{j}{2}$. Now, the fraction of busy periods that are under the influence of j - cold starts is:

$$\frac{s_j \cdot (j+1)}{\sum_{i=0}^{\infty} s_i(i+1)}$$

Unconditioning $\overline{D}(|j)$ gives:

$$\overline{D} = \frac{\sum_{j=0}^{\infty} \overline{D}(|j)\cdot s_j(j+1)}{\sum_{i=0}^{\infty} s_i(i+1)}$$

Substituting $\overline{D}(|j) = \frac{j}{2}$ gives:

$$\overline{D} = \frac{\overline{S} + \overline{S^2}}{2\cdot(1 + \overline{S})}$$

which agrees with our result.

From (2.13) we realize that $\overline{D}$ is monotonically increasing with $\overline{S}$, when $\overline{S^2}$ is held constant. Moreover, if instead we hold the squared coefficient of variation ($C_s^2 = (\overline{S^2} - (\overline{S})^2)/(\overline{S})^2$) fixed, and let $\overline{S}$ approach infinity, $\overline{D}$ will approach infinity too.

While all the previous properties look intuitive, the following is very surprising: $\overline{D}$ is not necessarily smaller than $\overline{S}$, i.e., the mean of the additional delay seen by a customer may be larger than the expected length of a cold start. Take, for example, the following cold start distribution:

$$s_i = \begin{cases} 1-\frac{1}{k} & i = 0 \\ \frac{1}{k} & i = k \\ 0 & \text{else} \end{cases}$$

So, $\overline{S} = 1$, $\overline{S^2} = k$. According to (2.13)

$$\overline{D} = \frac{2+(k-1)(1-x)}{2+2(1-x)}$$

Clearly, if $k > 3$, then $\overline{D} > 1$; so $\overline{D} > \overline{S}$!

Once this property is noted, the explanation is simple. The reason is that a short cold start affects only a few busy periods (in this extreme case, exactly one) and, therefore, only a few customers, while long cold starts affect many busy periods, and therefore many customers may see a large additional delay. Thus, if you take an average of all customers, the mean of the additional delay may exceed the average length of a cold start.

From this observation we realize that even if we hold $\bar{S}$ fixed, $\bar{D}$ can approach infinity when the second moment of the cold start is large enough. This observation is similar to the observation made about the delay suffered in an M/G/1 system. According to that observation (see for example, [Klei75]), the delay suffered in the M/G/1 system linearly increases with the coefficient of variation of the service time, so the delay may be unbounded even if ρ is kept fixed and under unity.

We conclude that the additional delay may grow extremely large if either the expected value of the cold start or the second moment of the cold start go to infinity.

2.3.4 The Eigenfunctions of the Discrete System

In the previous section we computed the additional delay D(z) suffered by the system customers as a function of the "cold start" length S(z). In this section we are interested in how the start-up delay distribution is transformed into the additional delay distribution. Mathematically, we may view equation (2.10) as a transformation from S(z) to D(z) and express it as:

$$D(z) \;=\; T\,(S(z)) \tag{2.14}$$

where T is the transformation expressed by (2.10).

We may now inquire as to what are the eigenfunction of this transformation. The mathematical meaning of this eigenfunction is: Find the solutions for the equation $S(z) = T(S(z))$. In other words, an eigenfunction of the system is an additional delay distribution (D(z)) which is identical to the cold start distribution (S(z)) causing it.

To solve for the eigenfunctions of our system, let us use (2.10) in (2.14):

$$S(z) \;=\; \frac{1}{1+(1-z)\bar{S}}\cdot\left[\frac{1-z+S(z)(z-z)}{1-z}\right] \tag{2.15}$$

Solving for S(z) gives:

$$S(z) = \frac{1-z}{(1-z)+\bar{S}(1-z)(1-z)}$$

or:

$$S(z) = \frac{\dfrac{1}{1+\bar{S}}}{1-\dfrac{\bar{S}z}{1+\bar{S}}} \tag{2.16}$$

Inverting (2.16) yields:

$$s_i = \frac{1}{1+\bar{S}} \cdot \left(\frac{\bar{S}}{1+\bar{S}}\right)^i \qquad i=0,1,2,\cdots \tag{17}$$

Yes! – the geometric distribution strikes again! As we already know, many "good things" in queueing theory have the memoryless property ...

In conclusion, then, if the cold start is geometrically distributed, the distribution of the additional delay suffered by *all customers* is also geometrically distributed with the same parameter.

Another important property of the eigenfunctions is that if the cold start is geometrically distributed (i.e., this is an eigenfunction of the system) then the additional delay is not a function of the system load. As a result, the expected value of the additional delay in a system where the starter is geometrically distributed, is *not affected* by the arrival rate.

2.3.5 The M/G/1 System with Bulk Arrivals - a Continuous Model

For the sake of completeness, we repeat the derivations made in the previous section for an M/G/1 system with bulk arrivals.

The system is a single-server system with exponential interarrival times (with parameter λ) and arbitrary service times. In this system arrivals have the memoryless property. Here the interarrival times are continuous, whereas they were previously discrete. As in the discrete case, the arrivals themselves may consist of bulks of arbitrary size. The derivations here are quite similar to our previous derivations. To make the reading easier, we will use the same equation indices (with little modification).

The basic notation is not changed: X_i, S_i , D_i have the same meaning as before, and equation (2.1) still holds. The probabilistic notation is the following:

$$D_i(t) \triangleq Pr[D_i \leq t], \qquad d_i(t) \triangleq \frac{\partial D_i(t)}{\partial t}, \qquad D_i^*(s) \triangleq \int_0^\infty e^{-st} d_i(t)dt,$$

$$X_i(t) \triangleq Pr[X_i \leq t], \qquad x_i(t) \triangleq \frac{\partial X_i(t)}{\partial t}, \qquad X_i^*(s) \triangleq \int_0^\infty e^{-st} x_i(t)xt,$$

$$S_i(t) \triangleq Pr[S_i \leq t], \qquad s_i(t) \triangleq \frac{\partial S_i(t)}{\partial t}, \qquad S_i^*(s) \triangleq \int_0^\infty e^{-st} s_i(t)st,$$

As in the previous section, the limits (when $i \to \infty$) of $d_i(t)$, $x_i(t)$, and $s_i(t)$ are denoted by d(t), x(t), and s(t), respectively. Similarly, $D^*(s)$, $X^*(s)$, $S^*(s)$ denote the limits of $D_i^*(s)$, $X_i^*(s)$, and $S_i^*(s)$. From the same arguments used in the previous section it is clear that $x(t) = x_i(t)$ and that $s(t) = s_i(t)$.

Since the arrival points form a Poisson arrival process, the interarrival times as well as the length of the idle periods are exponentially distributed (with parameter λ). Thus, we get:

$$X_i(t) = 1-e^{-\lambda t}, \qquad x(t) = x_i(t) = \lambda e^{-\lambda t}, \qquad X^*(s) = X_i^*(s) = \frac{\lambda}{\lambda+s} \tag{2.2b}$$

From (2.1) and corollary 2.4 we get:

$$d_{j+1}(t) = \int_{r=0^+}^{\infty} x(r)d_j(r+t)dr \; + \; s(t)\cdot\left(\int_{r=0}^{\infty} d_j(r)dr \int_{u=r}^{\infty} x(u)du\right) \tag{2.4b}$$

Taking Laplace transform of (2.4b) yields:

$$D_{j+1}^*(s) = \int_{t=0}^{\infty} e^{-st} \int_{r=0^+}^{\infty} x(r)d_j(r+t)drdt \; + \; \left[\int_0^{\infty} e^{-st}s(t)dt\right] \cdot \left[\int_0^{\infty} d_j(r)dr \int_{u=r}^{\infty} x(u)du\right] \tag{2.5b}$$

Substituting (2.2b) into (2.5b) and manipulating the expression gives:

$$D_{j+1}^*(s) = \frac{\lambda[D_j^*(\lambda)-D_j^*(s)]}{s-\lambda} \; + \; S^*(s)D_j^*(\lambda) \tag{2.6b}$$

Solving for $D^*(s)$ in equilibrium yields:

$$D^*(s) = D^*(\lambda)\cdot\left[\frac{\lambda+(s-\lambda)S^*(s)}{s}\right] \tag{2.7b}$$

Solving (2.7b) at $s=0$, using $D^*(0)=1$, $s^*(0)=1$ and L'Hospital's rule, we get:

$$1 = D^*(\lambda)\cdot\left[1 - \lambda\cdot\left.\frac{\partial S^*(s)}{\partial s}\right|_{s=0}\right] \tag{2.8b}$$

Using

$$\left.\frac{\partial S^*(s)}{\partial s}\right|_{s=0} = -\bar{S}$$

where $\bar{S}$ is the first moment of S, and we get:

$$D^*(\lambda) = \frac{1}{1+\lambda\bar{S}} \tag{2.9b}$$

Substituting into (2.7b), we obtain:

$$D^*(s) = \frac{1}{1+\lambda\overline{S}} \cdot \left[\frac{\lambda + S^*(s)(s-\lambda)}{s}\right] \qquad (2.10b)$$

As in the discrete case, here too, the Laplace transform of the *actual* delay suffered in the queue-with-starter can be calculated by multiplying the Laplace transform of the delay in the equivalent queue-without-starter by $D^*(s)$. This is true since these two variables are independent of each other.

From (2.10b) it is easy to derive $\overline{D}$:

$$\overline{D} = \frac{2\overline{S} + \lambda\overline{S^2}}{2 + 2\lambda\overline{S}} \qquad (2.13b)$$

The expressions for the Laplace transform of D (2.10) and for $\overline{D}$ (2.13) agree with Scholl's results [Scho76], which were calculated by a different method.

Now, to compute the eigenfunction of the system, we solve the equation:

$$S^*(s) = \frac{1}{1+\lambda\overline{S}} \cdot \left[\frac{\lambda + S^*(s)(s-\lambda)}{s}\right] \qquad (2.15b)$$

The solution is:

$$S^*(s) = \frac{1}{s\overline{S} + 1} \qquad (2.17b)$$

Inverting (2.17b) gives:

$$s(x) = \frac{1}{\overline{S}} \cdot e^{-\left(\frac{x}{\overline{S}}\right)} \qquad (2.18b)$$

which is, as expected, the exponential distribution!

2.4 An M/G/1 with Vacation Periods (Rest Periods)

Consider an M/G/1 system with unlimited storage. The arrival process is Poisson with arrival rate λ, and service order is first-come-first-served. When the server becomes idle, it goes for vacation of random length V. The probability density function of the vacation length is $v(t)$, and the Laplace transform of it is $V^*(s)$. If the server, upon returning from a vacation, finds any positive number of customers in the queue, it starts serving the customers as a regular M/G/1 (until the next vacation). If, on the other hand, the server finds no customers in the queue, it takes another vacation. Vacations are identically distributed and independent of each other and of arrival process or service times.

The M/G/1 system with vacation periods was first studied by Miller [Mill64] which analyzed, in addition to other system properties, the delay in the system. This system and similar ones were reported in [Levy75, Coop70, Heym77, Scho78, Shan80] and analyzed by different approaches. Scholl [Scho76] and Scholl and Kleinrock [Scho83] were the first to notice that the delay in an M/G/1 with vacation has the same distribution as a random variable which is the sum of two independent random variables:

- the time in system as if there were no vacation; plus

- an additional delay distributed as the residual life of the vacation period.

However, Scholl and Kleinrock [Scho83] emphasize that this is just an *observation* on the *expression* for the delay in the system with vacation. They were not able to show these properties directly (i.e., by analyzing the *system*).

In this section we show, in a direct way, that the additional delay in a system with vacation is independent of the delay in a system without vacation and that it is distributed as the residual life of the vacation distribution. First, using the queue-with-starter, we *directly* calculate the additional delay and find it to be as observed in [Scho83]. Second, we make a simple direct queueing analysis of the additional delay in the system with vacation and show that it is distributed as the residual life of the vacation.

2.4.1 Solving a System-with-Vacation by a System-with-Starter

In this analysis we notice that, as in the system-with-starter, additional delay is created only by a customer who arrives to an empty system. It is also clear that this customer is the first customer of some busy period as observed in system-A. Let us assume that this customer is C_k who arrives at time τ_k, and that the busy period in system-A, started by C_k, is the jth busy period. Since the additional delay suffered by C_k (and hence by all customers of busy period j) is due to the server's late return from vacation, let us call this delay *the return time* and denote it by R_j. In the following we show that the system with vacations can be considered as a system with starter where the start-up times (S_j) are the return-times (R_j). It is clear that in contrast to the cold starts the return times are *not independent* of all interarrival times. This is since a return time depends on the arrival process (for example, the return time R_j depends on the arrival epoch τ_k and therefore on the interarrival time t_k). For this reason all the theorems from section 2.3 have to be checked again, to make sure that they still hold when the cold starts are replaced by the return times. Even though the return times are not independent of all interarrival times, the following still holds:

THEOREM 2.6: The return time R_j is independent of all future interarrival times (t_{k+1}, t_{k+2},$\cdots$) and all future service times (c_k, c_{k+1},$\cdots$).

Proof: The return time R_j depends on the time (let us denote this time by t_0) at which the previous idle period (according to system-B) started, on the arrival time of C_k (which is τ_k), and on the vacations taken from t_0 until before τ_k. Since the vacation lengths (note, not their timing) are independent of each other and of system-A, all the above variables are independent of the sequences t_{k+1}, $t_{k+2}, \cdots$ and c_k, $c_{k+1}, \cdots$. Therefore, the return time, R_j, is also independent of these sequences. ■

In addition to theorem 2.6 we next show that in systems where the arrival process possesses the memoryless property the return time is also independent of the system history.

THEOREM 2.7: Let t_0 be the moment at which system-B becomes idle and the server starts taking vacations. Let j be the first busy period (according to system-A) starting after t_0. If the interarrival times possess the memoryless property (namely, they are geometrically distributed in discrete systems and exponentially distributed in continuous systems) then the return time R_j is independent of any property of system-B as observed prior to t_0.

Proof: It is clear that the return time R_j depends on the lengths of the vacation periods taken after t_0 and on the timing of the next arrival after t_0. Since the interarrival times possess the memoryless property, the time from t_0 to the next arrival is independent of the system history (prior to t_0). Since the vacation lengths are also independent of the system behavior, the return time is independent of the system behavior prior to t_0. ■

From theorems 2.6 and 2.7 it is now easy to realize that for an M/G/1 system all the theorems (and the analysis) from section 2.3 still hold if the cold-start times are replaced by the return-times. Therefore, the system with vacations can be considered as a system with starter where the role of the cold starts is played by the return times. For this reason, in the following we abandon the notations return-time and R_j and denote them, as done for the queue with starter, by cold starts and S_j.

The following corollary states that the M/G/1 with vacation periods can easily be solved by using the results of the M/G/1 with starter.

COROLLARY 2.8: An M/G/1 system with vacation periods can be solved as following:

1. Compute the distribution of a cold-start resulting from the vacation periods.

2. Use the expression for the Laplace transform of the cold-start distribution, computed in (2.1) above, and plug it into expression (2.10b).

3. The additional delay computed by this expression, is the additional delay in the system with vacation periods.

Next, we must calculate the distribution of a cold start. Keeping our old notation, we now add the vacation variable:

$V =$ the length of a vacation period;

$v(t) =$ the probability density function of V;

$V^*(s) =$ the Laplace transform of $v(t)$.

We recall that the length of a cold start is denoted by S and that of an idle period by X. Since the arrival process is Poisson with rate λ, $x(t) = \lambda e^{-\lambda t}$. Moreover, due to the memoryless property of the arrival process, any time interval which starts at an arbitrary point, t_0, and ends with the first arrival after t_0 is also exponentially distributed with parameter λ (like $x(t)$).

To calculate the length of a cold start, we start counting from the moment the system becomes idle; let us call this moment t_0. At t_0 the server goes on vacation. The time elapsing until the server returns is V. The first arrival after t_0 occurs X time units after t_0. If $X \leq V$, then the server, on returning from vacation, finds a customer in the system, and the additional delay that this customer will suffer is V-X. If, on the other hand, $X > V$, then the returning server will take another vacation. Again, due to the memoryless property of the arrival process, the first arrival will occur X time units after the end of the first vacation. Thus, if $X > V$ we can calculate the length of the cold start, recursively, as before.

All this explanation is summarized in the following recursion:

$$Pr[S \leq t] = \begin{cases} Pr[V\text{-}X \leq t] & \text{if } V \geq X \\ Pr[S \leq t] & \text{if } V < X \end{cases} \tag{2.18}$$

From this recursion we now solve for $S^*(s)$. From (2.18) and since V, X are independent:

$$s(t) = \int_{u=t}^{\infty} \lambda e^{-\lambda(u-t)} v(u)du \ + \ s(t) \cdot \int_{u=0}^{\infty} v(u) \int_{w=u}^{\infty} \lambda e^{-\lambda w} dw \ du$$

$$= \int_{u=t}^{\infty} \lambda e^{-\lambda(u-t)} v(u)du \ + \ s(t) \cdot V^*(\lambda) \tag{2.19}$$

Taking Laplace transform on (2.19):

$$S^*(s) = \int_{t=0}^{\infty} e^{-st} \int_{u=t}^{\infty} \lambda e^{-\lambda(u-t)} v(u) \ du \ dt \ + \ \int_{t=0}^{\infty} e^{-st} s(t) V^*(\lambda) dt \tag{2.20}$$

Inversion of the integration order on the left term yields:

$$S^*(s) = \int_{u=0}^{\infty} \lambda e^{-\lambda u} v(u)du \int_{t=0}^{\infty} e^{(\lambda-s)t} dt \ + \ V^*(\lambda) S^*(s) \tag{2.21}$$

$$S^*(s) = \frac{\lambda\cdot[V^*(s)-V^*(\lambda)]}{\lambda-s} + V^*(\lambda)S^*(s) \tag{2.22}$$

Solving for $S^*(s)$:

$$S^*(s) = \frac{\lambda[V^*(s)-V^*(\lambda)]}{(\lambda-s)[1-V^*(\lambda)]} \tag{2.23}$$

From (2.23) we now compute $\bar{S}$ by taking the derivative of $S^*(s)$ at $s=0$:

$$\frac{\partial S^*(s)}{\partial s} = \frac{\lambda}{1-V^*(\lambda)}\cdot\left[\frac{V^{*\prime}(s)(\lambda-s)+V^*(s)-V^*(\lambda)}{(\lambda-s)^2}\right]$$

Using $V^*(0)=1$, $V^{*\prime}(0)=-\bar{V}$, we get:

$$\left.\frac{\partial S^*(s)}{\partial s}\right|_{s=0} = \frac{\lambda}{1-V^*(\lambda)}\cdot\left[\frac{-\bar{V}\lambda+1-V^*(\lambda)}{\lambda^2}\right]$$

$$\bar{S} = \frac{1-\bar{V}\lambda-V^*(\lambda)}{\lambda\cdot(V^*(\lambda)-1)} \tag{2.24}$$

Now that we know the Laplace transform and the first moment of the starter distribution, we can compute the Laplace transform of the additional delay using equation (2.10b) from the analysis of the queue with starter. Recall equation (2.10b):

$$D^*(s) = \frac{1}{1+\lambda\bar{S}}\cdot\left\{\frac{\lambda+S^*(s)(s-\lambda)}{s}\right\}$$

We now substitute (2.23), (2.24) into (2.10b) to obtain the additional delay for a system with vacations:

$$D^*(s) = \frac{1-V^*(s)}{\bar{V}\cdot s} \tag{2.25}$$

Yes! this is the residual life of the vacation period! We have thus shown that the additional delay in an M/G/1 system with vacation periods is independent of the original delay and is distributed as the residual life of the vacation period.

2.5 Direct Explanation for the Delay of a Queue with Vacation

In the previous section we showed by direct calculation that the delay in a queue with vacation actually <u>is</u> (and not only "could be thought as") the sum of two independent random variables:

- the delay in a queue without vacation;
- additional delay distributed as the residual life of the vacation period;

yet we did not give a direct queueing explanation for the fact that the additional delay is distributed as the residual life of the vacation period. We shall do that in this section.

Consider the busy and idle periods in a regular M/G/1 system (denoted as system-A) as described in figure 2.2a. We denote busy periods by Y_1, Y_2, $\cdots$ and idle periods by X_1, X_2, $\cdots$. Now, let us impose vacations on this system (the new system is denoted as system-B). For "pedagogical" reasons, let us assume that the "vacation" is just another job the server has to do. Thus, if we look from the *Server's point of view* we notice three properties:

1. The server always consumes work at rate of "one unit of work per unit of time."

2. At time points where a vacation V_i starts, additional work, equaling (in amount) the vacation length $|V_i|$, arrives at the system (remember, from the server's point of view!).

3. A new vacation starts *if and only if* the amount of work in the system is exactly zero. This means that the server takes a new vacation either when it finishes working in the M/G/1 system or when it returns from vacation and finds the M/G/1 system still empty of customers.

This situation is illustrated in figure 2.2b. The solid line represents the total amount of work as seen by the server (denoted by $U_{server}(t)$), while the broken line represents the unfinished work in the M/G/1 system with no vacations (denoted by $U_{M/G/1}(t)$). We notice that the server is continuously busy at a rate of "one work unit per time unit" ($\rho=1$) and that "vacation work" always arrives to the server system when $U_{server}(t)$ drops to zero. Next, we notice that the server system serves in first-come-first-served (FCFS) order. This is because of the following properties:

- M/G/1 customers are served according to FCFS policy;

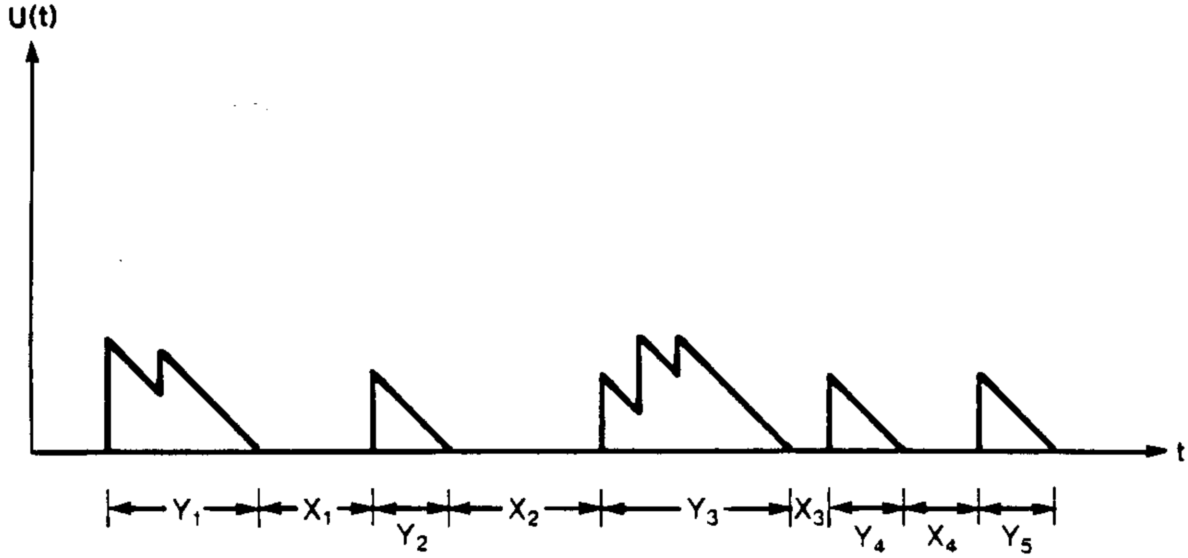
- "vacation customers" arrive only when there are no M/G/1 customers;

- the server completes service of any customer of any type before serving the next customer (nonpreemptive system).

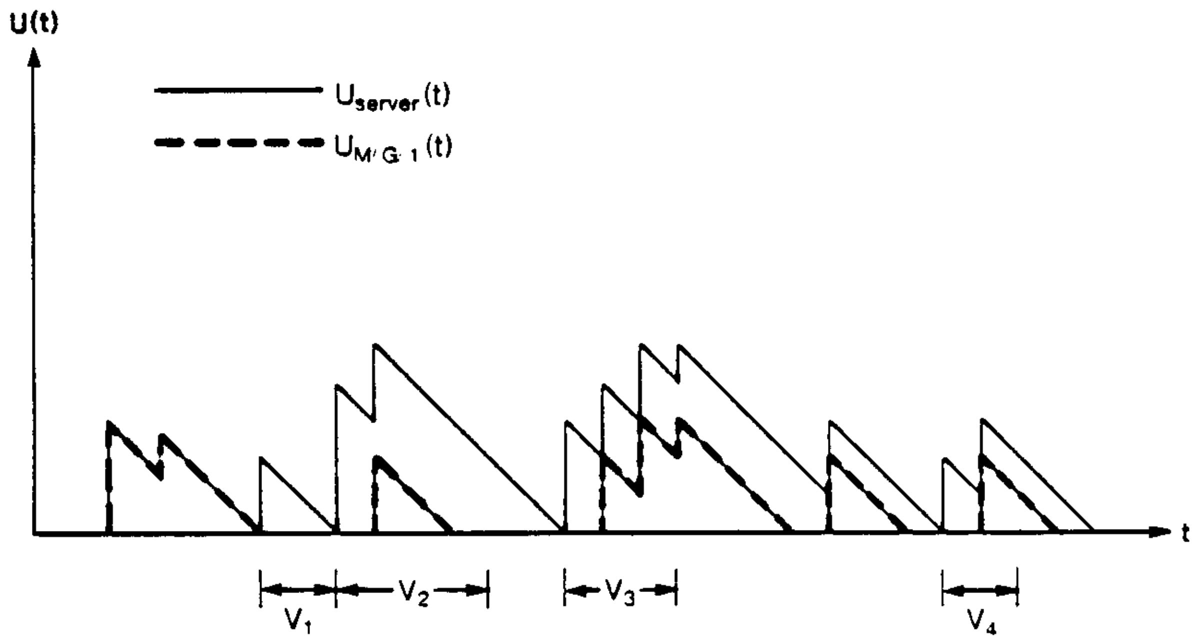
For this reason it is clear that the total time in system for an M/G/1 customer arriving at time t to the system with vacation is exactly $U_{server}(t)$. Clearly, the time in system for the same customer in a system without vacations is $U_{M/G/1}(t)$; thus, the additional delay suffered by this customer is given by $U_{server}(t) - U_{M/G/1}(t)$.

In figure 2.3a we plot the function difference $U_{server}(t) - U_{M/G/1}(t)$ (denoted by $D(t)$) versus t. In this figure the following properties can be noticed:

1. In time segments corresponding to idle periods in system-A (figure 2.2a), $D(t)$ is consumed at the rate of "one work unit per time unit."

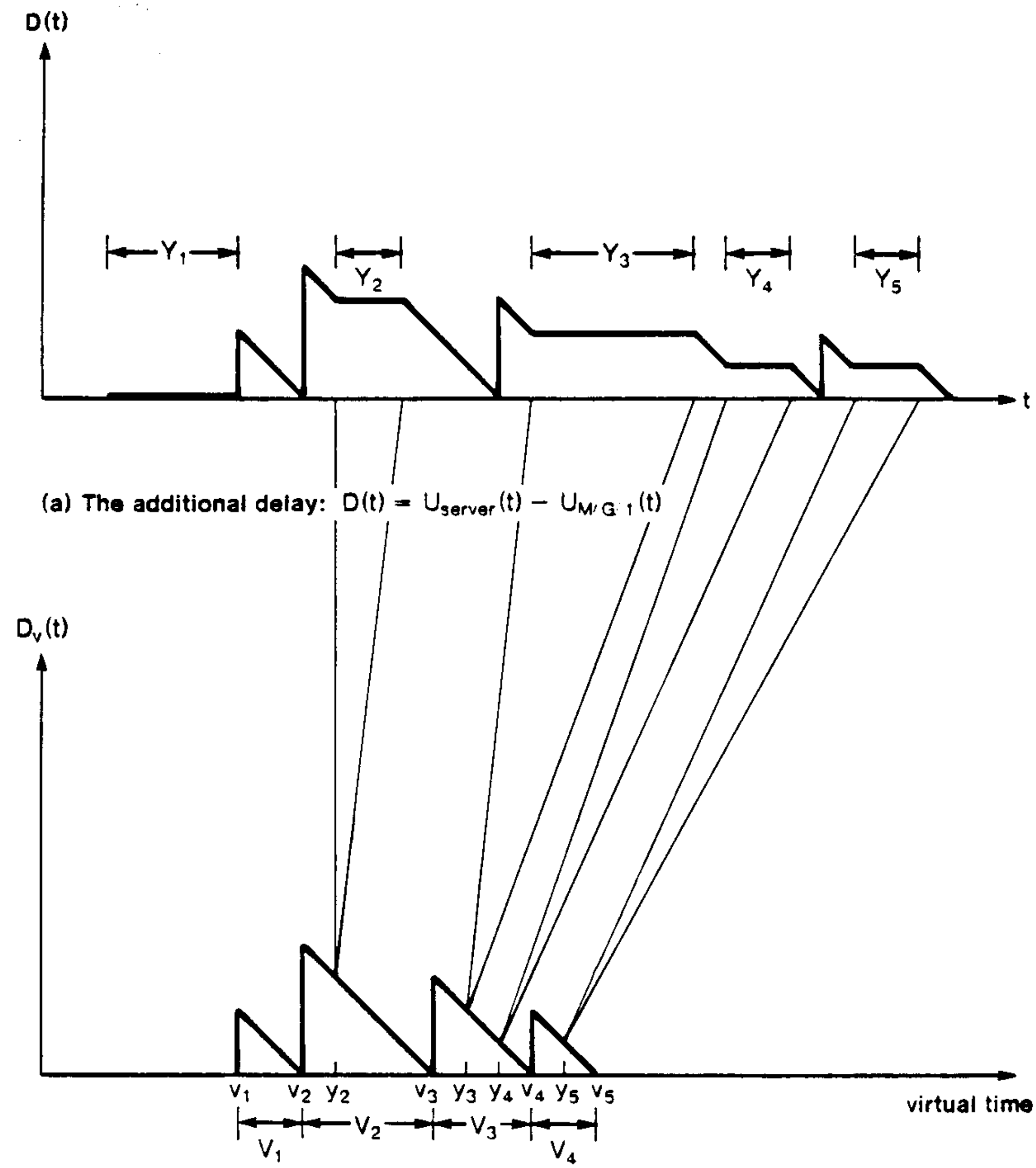


(a) The unfinished work in a regular M/G/1



(b) Vacation periods "added" to a regular M/G/1

Figure 2.2: The unfinished work in a system with vacation periods



(b) The additional delay versus virtual time

Figure 2.3: The additional delay in a system with vacations

2. In time segments corresponding to busy periods in system-A, $D(t)$ remains constant.

3. The time epochs where $D(t)$ increases are those corresponding to the beginning of vacations. At such a moment, $D(t)=0$ and discontinuously increases to the "height" of the vacation starting at that time.

In this figure we note that $D(t)$ is independent of any property of a system-A busy period (excluding its timing) since it stays constant during the duration of such periods. $D(t)$ is determined only by the *length of the vacations* and the *length of the system-A idle periods*. This is the reason why the additional delay in the queue with vacation (as in the queue with starter) is independent of the delay in the regular M/G/1 system.

Since additional delay is independent of any property of system-A busy periods (excluding timing), we can represent any system-A busy period by its starting point only. We do so by contracting the flat segments of $D(t)$ to a point. This is done in in figure 2.3b, where the time axis becomes a virtual time axis and a segment Y_i from figure 2.3a is contracted to a point y_i. The point corresponding to the beginning of a vacation, V_i, is denoted by v_i. For this figure we define $D_v(t)$ as the (virtual) additional delay of virtual time t as seen in figure 2.3b. In the transformation from 2.3a to 2.3b, we notice the following properties:

1. $D_v(y_i)$ equals the additional delay suffered by all customers of busy period (in 2.2a) Y_i.

2. D_v continuously decreases at the rate of "one work unit per time unit." Whenever D_v becomes zero, it increases by a discontinuous increment.

3. The increments of D_v occur in epochs corresponding to vacation starts. The increment size is the vacation length.

4. Let t be an arbitrary time epoch on the virtual time axis and v_i be the epoch corresponding to the first vacation starts after t. From properties 2 and 3 and from the structure observed in figure 2.3b it is clear that $D_v(t)=v_i-t$.

From these arguments it becomes clear that, in order to find the additional delay suffered in system-A, one may compute D_v for the points $\{y_i\}$ in figure 2.3b. This can be done as follows: We take a large time segment $(0,t)$ in figure 2.3b and examine $D_v(y_i)$ for all y_i in this segment. We first note that the length of a subsegment (v_i,v_{i+1}) is distributed according to the distribution of the vacation length. Then, we notice that the intervals between the adjacent y-points represent lengths of idle periods; so, they are exponentially distributed with parameter λ. Thus, in 2.3b the y-points behave like a stream of Poisson arrivals. Now, let n be the number of y-points in $(0,t)$. From the property of the Poisson distribution (see, for example, [Klei75]), the intervals between the y-points have the same statistics as if these points were selected from a uniform distribution on $(0,t)$. Suppose that this was the way that the y-points were created, and let us examine what $D(y_i)$ is. Things now become clear: We randomly, according to the uniform distribution, drop a point y_k on the segment $(0,t)$. This segment is divided to subsegments, $(v_1,v_2),(v_2,v_3),...$ Thus, the time difference between a point y_k and the next point v_j , $D_v(y_k)$, is the residual life of the segments $\{(v_i,v_{i+1})\}$. Since the lengths of the segments $\{(v_i,v_{i+1})\}$ are distributed as the vacation lengths, we get $D_v(y_k)$ distributed as the residual life of the vacation period.

Thus, we arrive at the expected conclusion: $D_v(y_i)$, the additional delay of the customers in a system with vacation periods, is distributed as the residual life of a vacation period!

2.6 Summary

The queueing system with an independent starter and the queueing system with independent vacation periods were studied in this chapter. It was shown that the delay distribution in the queue with starter is composed of the direct sum of two independent variables: 1) The delay in the equivalent queue without starter. 2) The additional delay suffered due to the starter presence. Using this decomposition property, the Laplace transform of the additional delay was derived. This transform was derived both for discrete time systems with geometrically distributed interarrival times and for continuous time systems with Poisson arrivals. Using the same approach, we then analyzed the M/G/1 system with vacation periods. It was first shown that the M/G/1 with vacations is just a special case of the M/G/1 with starter, so that the delay in the M/G/1 with vacations can be easily found by using the formula for the delay of the M/G/1 with starter. Second, using geometric arguments we explained why the additional delay in the vacation system is distributed as the residual life of the vacation period.

CHAPTER 3

A Queue with an Non Independent Starter: Delay Analysis

A queueing system in which a start-up delay is incurred whenever an idle period ends is analyzed in this chapter. In chapter 2 this system was analyzed under the assumption that the start-up delay is a random variable *independent* of the system behavior. In this chapter we extend the study of the "queue with starter" and allow the start-up delay to depend on the system behavior. Two types of systems are analyzed: 1) A system where the start-up delay depends on the amount of work (or the number of customers) arriving to the system at the beginning of the start-up period. 2) A system where the start-up delay depends on the length of the idle period preceding the start-up operation. The analysis is done for systems (both discrete and continuous) where the interarrival periods possess the memoryless property. The results of this analysis are used in chapter 5 in the analysis of some exhaustive slotted ALOHA systems.

3.1 Introduction and Previous Work

In the following we consider a queueing system with a "starter." In such a system the server is "turned off" whenever it becomes idle. When a customer arrives at an idle system, it cannot be served immediately; rather, an additional (random) amount of time is required to start the "cold" system before the new "first" customer can be served. Customers who arrive at a "hot" system (i.e. one with at least one customer in the system) will join the queue and be served in turn.

The queue with starter has been studied before under the basic assumption that the length of the starter (or, the length of the set-up time) is a random variable *independent* of the system behavior; for a literature review of the independent-starter system, see chapter 2. In this chapter we study a new variation of the queue with starter. In contrast to the previous studies and to the analysis carried in chapter 2 we *do not assume* that the starter behavior is independent of the system status. Rather, we allow the set-up time to depend on two system factors:

1. The amount of work brought to the system by the new arriving customer(s).

2. The length of the time the system was idle prior to the set-up period. Namely, the length of the idle period preceding the starter operation.

To understand the importance of these new variations for the modeling of queueing systems, let us describe what systems can be modeled by the new models, compared to the systems that can be modeled by the old independence model:

1. The queue with an *independent starter* can be described as a system with an "indifferent" server. In this system the server goes for a "vacation" of indefinite length whenever the system gets idle and returns to service when new customer(s) arrives. However, the return of the server is not instantaneous since it takes the server a random amount of time to complete his vacation (or just to start working again). We call this type of server "the indifferent server" since the time it takes the server to start working is independent of the system behavior.

2. A queue with starter where the set-up time depends on the amount of work brought to the system by the new arriving customer, can model a system with customer-sensitive server. In this system the server is sensitive to the amount of work brought by the first customer (or to the number of customers arriving in the first bulk, in a bulk-arrival system) and the return-time depends on this amount. In many real queueing systems it is only natural that the server will "rush" back to work when it sees a "big customer" arriving (much work to do...) and will be much slower if the new arriving customer is just a "small one". Clearly, this is not the only application that can be modeled by a starter that depends on the amount of work brought to the system. As a matter of fact, as already stated, the analysis of the queue with a non independent starter was motivated by the need to analyze some exhaustive slotted ALOHA systems, and the results derived in this chapter are used in the analysis of those ALOHA systems (chapter 5).

3. The queue with starter which depends on the length of the last idle period can be used to model systems where the server is some machine which is turned-off whenever the system gets empty, and the set-up time is a "warm-up" period of the machine. As we know from behavior of automobiles, this warm-up period may directly depend on the amount of time the machine was idle before starting-up: the longer the idle period, the longer the warm-up time.

In section 3.2 we review the model description and the analysis of the queue with an independent starter reported in chapter 2. Using the analysis approach and the results reported in chapter 2 it is very convenient to analyze the queue with *non independent* starter. The properties of the queue with an independent starter which are important for this chapter are reviewed in section 3.2. Also reviewed in section 3.2 are the model description and the notation used in chapter 2.

Section 3.3 deals with a discrete (slotted) system with memoryless bulk arrivals, where the start-up delay depends on the number of customers arriving in the first bulk after an idle period. For this system we derive the z-transform and the expected value of the additional delay an arbitrary customer suffers in the system. We realize that the expected value of the additional delay (unlike the z-transform) can be used directly to calculate the expected value of the total delay suffered in the system.

In section 3.4 we are interested in a continuous time system. Here the underlying system is an M/G/1 system and the starter depends on the amount of work brought to the system by the first customer of a busy period. The analysis in this section is similar to that done in section 3.3 and it yields the Laplace-Stieltjes transform (LST) and the expected value of the additional delay suffered by an arbitrary customer in the system. As in section 3.3 we realize that the expected value of the additional delay can be used directly to calculate the expected value of the total delay suffered in the system. Nevertheless (again, as in section 3.3), the LST of the additional delay *cannot be used directly* to calculate the LST of the total delay suffered in the system. This is true since the additional delay suffered by an arbitrary customer in the system with starter is *not independent* of the delay this customer would suffer in the system without starter.

The LST of the total delay suffered by an arbitrary customer in this type of system, is the topic of section 3.5. Due to the non-independence property, observed above we must go through a more complicated analysis in order to derive this LST.

In section 3.6 we deal with a queue with starter where the starter depends on the length of the time the system was idle before the start-up time. It is observed that for systems with memoryless arrivals the dependence of the starter on the system behavior is quite simple, so using the results reviewed in section 3.2 it is rather trivial to calculate the additional delay suffered in this system. This result has been previously reported in [Welc64] and is discussed here for completeness. Section 3.6 also deals with mixed systems, namely, systems where the starter depends both on the length of the preceding idle period and on the amount of work brought to the system by the first customer in the busy period.

Section 3.7 deals with extensions and generalizations of the basic models. It is observed that most of the results reported in the previous sections can easily be extended (sometimes by a small modification) to the following types of systems: 1) Systems with bulk arrivals. 2) Systems where the starter depends on the size of the "first bulk". 3) Systems where the starter depends on the amount of work brought to the system by the "first bulk".

3.2 Model Description, Notation and Review of the Queue with Independent Starter

In this section we review the results for a queue with starter where the start-up times are independent of the other system parameters. These results were reported in chapter 2 and will be used throughput this chapter.

The model considered here is simple: We consider a single server queue in which a customer that arrives to an empty system cannot be served right away. Rather, an additional (random) amount of time is required to start the system up, before this customer can be served. This period of time, during which the server "warms up" is called *a cold start*. The lengths of the cold starts in a system with *an independent starter* are identically distributed and are *independent* of any of the system parameters.

The notation used and the assumptions on the arriving customers are the following:

1. Arrivals occur at time epochs $\tau_1, \tau_2, \cdots$. The arrivals may consists of a bulk of customers (in several of the cases studied) or of single customers. Interarrival times are denoted by $t_1, t_2, \cdots$. Thus, $t_i = \tau_i - \tau_{i-1}$.

2. The interarrival times are statistically independent of each other. Namely, t_i is independent of t_j for $i \neq j$.

3. The ith customer* is denoted by C_i.

4. The service time of C_i is denoted by c_i.

5. The service times are statistically independent of each other. Namely, c_i is independent of c_j for $i \neq j$.

6. The interarrival times are statistically independent of the service times. Namely, t_i is independent of c_j for every i and j.

The notation and the assumptions made above on the arrivals are valid both for the systems with an independent starter and for the systems with a non-independent starter.

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*Note that some ambiguity in the numbering of the customers and of the arrivals may arise in the case of bulk arrivals. However, since the actual service times are not required in the analysis, this ambiguity is avoided.

To analyze this system we compare it to a similar system (under the same arrival realization) which does not suffer cold starts. We call the system with starter, *system-B,* and the corresponding system that does not have a starter, *system-A.* The analysis is done by observing the *unfinished work,* denoted by $U(t)$, in the two systems.

The system time suffered by a customer in system-B is calculated by comparing it to system-A. We call the time spent by a customer in system-B, *the total delay* (or *the system time*). This delay is viewed as the sum of two different delays: a) The delay this customer would suffer in the equivalent system-A, which we call *the original delay.* b) The additional delay this customer suffers due to the starter "presence", called *the additional delay.*

In figure 2.1, we plotted the unfinished work as observed in the two systems. Figure 2.1a describes $U(t)$ in system-A, and figure 2.1b describes $U(t)$ in system-B. In this figure, Y_i denotes the ith busy period according to the realization in system-A, and X_i denotes the corresponding ith idle period. D_i denotes the additional delay suffered in system-B by the customers of the ith busy period (busy period are indexed by their realization in system-A).

The following facts were established in chapter 2:

1. The additional delay suffered by all customers of the ith busy period is the same. Thus, D_i is the additional delay suffered by *all* customers of the ith busy period.

2. The additional delay suffered by an arbitrary customer in system-B is independent of the delay it suffers in system-A.

3. A busy period (b.p.), Y_i, may "suffer" a cold start (as do Y_1 and Y_2); in this case the additional delay suffered by the customers of this b.p. is distributed as the distribution of the cold start. However, not all busy periods do suffer cold starts. In this case the additional delay suffered by the jth b.p. is some residual of the additional delay suffered by the j-1*st* b.p.; for example see that the additional delay suffered by the customers of Y_3 is some residual of the additional delay suffered by the customers of Y_2.

Using the above observations one can recursively calculate the additional delay suffered by the customers of the ith busy period from the additional delay suffered by the customers of the i-1st busy period. From this recursion it is then easy to derive the Laplace-Stieltjes transform (LST) of the additional delay suffered by an arbitrary customer when the system is in equilibrium. Since the additional delay is independent of the original delay, this LST can be multiplied by the LST of the original delay suffered in the corresponding system-A, to yield the LST of the total delay suffered in system-B.

In chapter 2 we analyzed the queue with an independent starter for two types of systems: 1) A discrete system where the arrivals consist of bulks of customers and the interarrival times are geometrically distributed (with parameter x). 2) A continuous system with Poisson arrivals (an M/G/1 system).

The notation used for analyzing the discrete system are the following:

$S_j \triangleq$ the length of the cold start (if any) suffered by the jth busy period

$D_j \triangleq$ the additional delay suffered by the jth busy period

$S_j(z) \triangleq$ the z-transform of the distribution of S_j

$D_j(z) \triangleq$ the z-transform of the distribution of D_j

Similarly, $S(z)$ and $D(z)$ denote the z-transforms of the cold start and of the additional delay suffered by (customers of) an arbitrary busy period when the system is in equilibrium. In addition $\overline{S}$, $\overline{S^2}$ denote the first and second moments of the cold start distribution and $\overline{D}$ denotes the equilibrium expected value of the additional delay, suffered by an arbitrary busy period. x denotes the probability of no arrival to the system in a given slot.

The recursive relation of the additional delay in this system was found to be:

$$D_{j+1}(z) = \frac{(1-x)\cdot[D_j(x)-D_j(z)]}{x-z} + S(z)\cdot D_j(x) \tag{3.1}$$

The z-transform of the additional delay suffered by an arbitrary busy period, when the system is in equilibrium, is:

$$D(z) = \frac{1}{1+(1-x)\overline{S}}\cdot\left[\frac{1-x+(x-z)S(z)}{1-z}\right] \tag{3.2}$$

The expected additional delay is given by:

$$\overline{D} = \frac{2\overline{S}+(\overline{S^2}-\overline{S})(1-x)}{2+2\overline{S}(1-x)} \tag{3.3}$$

The notation used for analyzing the continuous system (M/G/1) are the following:

$S_j^*(s) \triangleq$ the LST of the distribution of S_j

$D_j^*(s) \triangleq$ the LST of the distribution of D_j

where S_j and D_j have the same meaning as in the discrete system.

Similarly, $S^*(s)$ denotes the LST of the cold start distribution and $D^*(s)$ denotes the LST of the additional delay suffered by an arbitrary busy period under equilibrium. $\overline{S}$, $\overline{S^2}$ and $\overline{D}$ have the same meaning as in the discrete system. λ is the parameter of the Poisson arrivals.

The recursive relation of the additional delay in this system is given by:

$$D_{j+1}^*(s) = \frac{\lambda[D_j^*(\lambda) - D_j^*(s)]}{s - \lambda} + S^*(s)D_j^*(\lambda) \tag{3.4}$$

The LST of the additional delay suffered by an arbitrary busy period is:

$$D^*(s) = \frac{1}{1+\lambda\overline{S}}\left[\frac{\lambda + S^*(s)(s-\lambda)}{s}\right] \tag{3.5}$$

The expected additional delay is given by:

$$\overline{D} = \frac{2\overline{S} + \lambda\overline{S^2}}{2 + 2\lambda\overline{S}} \tag{3.6}$$

Having reviewed the queue with independent starter we now turn to analyze queueing systems with starter where the length of the cold start is not independent of the system behavior. Considering the arrival process, these systems are similar to the queue with independent starter. The difference is that in the non-independent systems the length of the cold start is not chosen independently of the arrival process. Rather, the length of a cold start is a function of this process.

3.3 The Discrete System Where the Starter Depends on the Number of Customers it finds in the System

Our first system is a discrete queueing system with starter. The arrival process in this system consists of bulks of arbitrary size. Therefore, a system-A busy period starts with the amount of work brought by the "first" bulk (we will refer to the customers in the first bulk as "first" customers). In contrast to chapter 2, we do not assume here that the length of a cold start is independent of the amount of work brought by these first customers. Rather, the length of the cold start is a function of the *number of customers* in this first bulk. This dependency is denoted by the following notation:

$s_i^j|_k \triangleq Pr[S_j = i$ | the number of customers starting busy period j is $k]$

$$S_j(z|k) \triangleq \sum_{i=0}^{\infty} s_i^j|_k \cdot z^i \ , \quad \overline{S_j|_k} \triangleq \sum_{i=0}^{\infty} i \cdot s_i^j|_k$$

To avoid confusion, note that this notation is valid only for a busy period j that suffers a cold start (in contrast to many busy periods that do not suffer a cold start). The conditional z-transform of an arbitrary cold start (without considering the busy period index) and its expected value are denoted by $S(z|k)$ and $\overline{S|_k}$, respectively. Clearly, since we assume that the distribution of a cold start is independent of the busy period index, $S(z|k) = S_j(z|k)$ and $\overline{S_j|_k} = \overline{S|_k}$ for every k.

In addition we define:

$a_i \triangleq Pr[$ bulk consists of i customers $]$; $i=0,1,2,\cdots$

$b_i \triangleq Pr[$ a system-A busy period starts with a bulk of i customers $]$; $i=0,1,2,\cdots$

$$\overline{b} \triangleq \sum_{i=1}^{\infty} i \cdot b_i$$

$x_i \triangleq Pr[$ the length of an arbitrary idle period is i time units $]$

From this notation, it is clear that $b_i = \dfrac{a_i}{1-a_0}$ for $i \neq 0$. In addition, if we denote by x the probability of no arrival in a given slot ($x = a_0$), we can easily calculate x_i:

$$x_i = (1-x) \cdot x^{i-1} \qquad i=1,2,3,...$$

Using this notation we may write:

$$S_j(z) = \sum_{k=1}^{\infty} S_j(z|k) \cdot b_k \ , \quad S(z) = \sum_{k=1}^{\infty} S(z|k) \cdot b_k \ , \quad \overline{S} = \sum_{k=1}^{\infty} \overline{S|_k} \cdot b_k$$

This allows us to calculate the unconditional expression for the length of the cold start from the conditional expressions.

Similarly, we denote the additional delay suffered by customers of the jth busy period, conditioning on the size of the first bulk of this busy period:

$d_i^j|_k \triangleq Pr[D_j = i$ | the number of customers starting busy period j is $k]$

$$D_j(z\,|k) \triangleq \sum_{i=0}^{\infty} d^j_{i|k} z^i$$

The additional delay suffered by an *arbitrary* customer is denoted by:

$g_i \triangleq Pr[$ the additional delay suffered by an arbitrary customer is $i\,]$

The z-transform of this distribution, and its expected value are denoted by:

$$G(z) \triangleq \sum_{i=0}^{\infty} g_i z^i \quad , \quad \overline{G} \triangleq \sum_{i=0}^{\infty} i g_i$$

We may now start deriving the additional delay suffered by customers in the system under equilibrium conditions. First, let us pick an arbitrary customer and calculate the additional delay suffered by this customer. Without loss of generality, we can assume that the system-A busy period in which this customer is served, is $j+1$. Next, let us condition the delay suffered by this customer on the number of customers (k) that start busy period $j+1$. As shown in chapter 2 the additional delay suffered by this customer depends on the additional delay suffered by the customers of the jth busy period, on the length of the jth idle period and on the length of the cold start (if any) starting the $j+1st$ busy period. As in the system with an independent starter, here too, these three measures are *independent* of each other. Thus, the additional delay of our customer is:

$$d^{j+1}_{i|k} = \sum_{m=1}^{\infty} x_m d^j_{m+i} + s^{j+1}_{i|k} \cdot \sum_{m=0}^{\infty} d^j_m \sum_{l=m+1}^{\infty} x_l \tag{3.7}$$

The first term of this expression represents the situation where the additional delay suffered by busy period $j+1$ is the residual of the additional delay suffered by the jth busy period. The second term represents the situation where the additional delay suffered by the jth busy period is smaller than the length of the jth idle period. In this situation busy period $j+1$ suffers the additional delay caused by a cold start.

Taking the z-transform of (3.7) and using the fact that the distribution of the cold start is independent of its index, we get:

$$D_{j+1}(z\,|k) = \frac{(1-x)\cdot[D_j(x)-D_j(z)]}{x-z} + S(z\,|k) \cdot D_j(x) \tag{3.8}$$

Since the system is in equilibrium, the additional delay suffered by the customers of the jth busy period is just the equilibrium additional delay. We observe that this additional delay is distributed exactly as the additional delay in a queue with independent starter. Thus, using the derived unconditional expression for $S(z)$ (see section 3.2 equation (3.2)) we have:

$$D_j(z) = D(z) = \frac{1}{1+(1-x)\overline{S}} \cdot \left[\frac{1-x+S(z)(x-z)}{1-z} \right] \tag{3.9}$$

Thus, replacing $D_j(z)$ for $D(z)$ in equation (3.8) we have:

$$D_{j+1}(z \mid k) = \frac{(1-x)\cdot[D(x)-D(z)]}{x-z} + S(z \mid k) \cdot D(z) \tag{3.10}$$

Next, let us compute the probability that the busy period to which our arbitrary customer belongs, starts with k customers. It is clear that the expected number of customers served in a busy period which was started by k customers is proportional to k. This is true since this busy period can be thought of as k consecutive independent sub-busy periods, each of which starts with one customer (see for example [Klei75], section 5.8). From this argument it is easy to calculate the probability that an arbitrary customer belongs to a busy period that starts with k customers:

$$Pr[\text{ arbitrary customer belongs to a busy period started by } k \text{ customers }] = \frac{b_k \cdot k}{\sum_{i=1}^{\infty} b_i \cdot i} \tag{3.11}$$

Now, it is clear that equation (3.10) is actually independent of the busy period index ($j+1$). Thus if we condition on the fact that our arbitrary customer is served in a busy period started by k customers, then the z-transform of the additional delay suffered by this customer is given by:

$$D(z \mid \text{busy period starts with } k \text{ customers}) = \frac{(1-x)\cdot[D(x)-D(z)]}{x-z} + S(z \mid k) \cdot D(z) \tag{3.12}$$

From equation (3.11) and equation (3.12) it is now easy to calculate the z-transform of the additional delay suffered by an arbitrary customer. This is done by unconditioning equation (3.12):

$$G(z) = \sum_{k=1}^{\infty} \frac{b_k k}{\sum_{i=1}^{\infty} b_i i} \cdot \left[\frac{(1-x)\cdot[D(x)-D(z)]}{x-z} + S(z \mid k) \cdot D(z) \right]$$

$$G(z) = \frac{(1-x)\cdot[D(x)-D(z)]}{x-z} + D(z) \cdot \frac{\sum_{k=1}^{\infty} S(z \mid k) b_k k}{\overline{b}} \tag{3.13}$$

Using equation (3.9) we may rewrite this as:

$$G(z) = D(z) + D(z) \cdot \left[\frac{\sum_{k=1}^{\infty} S(z \mid k) b_k k}{\overline{b}} - S(z) \right] \tag{3.14}$$

from which we observe that the z-transform of the additional delay in our system equal to the additional delay in the system with an independent starter plus the second term of (3.14). Now substituting D(z) from equation (3.2) gives:

$$G(z) = \frac{1}{1+(1-z)\overline{S}} \cdot \left[\frac{(1-z)(1-S(z))}{1-z} + \frac{\sum_{k=1}^{\infty} S(z\ |k) b_k k}{\overline{b}} \right] \tag{3.15}$$

Note, that neither $G(z)$ from equation (3.15) nor $D(z)$ from equation (3.9) can be used directly to solve for the z-transform of the *total* delay to a customer in the queue-with-starter. Unlike the approach suggested in chapter 2 for a queue with an independent starter, we cannot compute this z-transform by multiplying any of the above transforms by the z-transform of the queue-without-starter. This is true, since in this system the additional delay in the queue-with-starter is *not independent* of the delay suffered in the queue-without-starter. Nevertheless, if one is interested in the *expected value* of the total delay, one can compute it by summing the expected additional delay and the expected delay in the queue-without-starter. The expected value of the additional delay in the system is calculated from (3.14):

$$\overline{G} = \overline{D} + D(z) \cdot \left[\frac{\sum_{k=1}^{\infty} \overline{S_{|k}} \cdot b_k k}{\sum_{k=1}^{\infty} b_k k} - \overline{S} \right] \tag{3.16}$$

or, by using (3.3):

$$\overline{G} = \frac{1}{1+(1-z)\overline{S}} \cdot \left[\frac{(\overline{S^2} - \overline{S})(1-z)}{2} + \frac{\sum_{k=1}^{\infty} \overline{S_{|k}} \cdot b_k k}{\sum_{k=1}^{\infty} b_k k} \right] \tag{3.17}$$

3.4 The M/G/1 System where the Starter Depends on the Amount of Work Brought by a First Customer: Analysis of the Additional Delay

In this section we are interested in the analysis of the additional delay in an M/G/1 system with a non-independent starter. The system model is the following: the basic system is a regular M/G/1 queue. The modification is that when a customer arrives to an empty system a random amount of time is required to "start" the server before this customer can be served. This amount of time *depends* on the amount of work brought to the system by that "first customer".

The goal here is to analyze the *additional-delay* suffered in the system. Thus, as in section 3.3, we will derive the LST and the expected value of the additional delay suffered by an arbitrary customer in the system. As in the discrete system, one can sum the expected value of the additional delay in the system and the expected value of the system time in the corresponding M/G/1, to yield the *expected value of the system time*. On the other hand, the LST of the additional delay suffered in the system cannot be used in a direct way (namely, by multiplying it by the LST of the system time in the corresponding M/G/1) to calculate the total delay in the system. Nevertheless, using more sophisticated analysis and the results derived in this section, we will be able, in the next section, to derive the LST of the total delay suffered in the system.

The notation to be used is similar to that used in section 3.2, but here we condition on the service time of the first customer of the given busy period:

$$D_i(t \mid z) \triangleq Pr[D_i \leq t \mid \textit{the service time of the first customer in b.p. } i = z]$$

$$d_i(t \mid z) \triangleq \frac{\partial D_i(t \mid z)}{\partial t}, \qquad D_i^*(s \mid z) \triangleq \int_0^\infty e^{-st} d_i(t \mid z)dt,$$

$$S_i(t \mid z) \triangleq Pr[S_i \leq t \mid \textit{the service time of the first customer in b.p. } i = z]$$

$$s_i(t \mid z) \triangleq \frac{\partial S_i(t \mid z)}{\partial t}, \qquad S_i^*(s \mid z) \triangleq \int_0^\infty e^{-st} s_i(t \mid z)dt \qquad \overline{S_i \mid_z} \triangleq \int_0^\infty t \cdot s_i(t \mid z)dt,$$

The conditional LST and the conditional expected value of the cold start suffered by an arbitrary busy period (disregarding its index) are denoted by $S^*(s \mid z)$ and $\overline{S \mid_z}$ respectively. The conditional LST of the additional delay suffered by an arbitrary busy period is denoted by $D^*(s \mid z)$. The unconditioned cold start is denoted by $S^*(s)$, $S(t)$, s(t), and $\overline{S}$. The unconditioned additional delay (suffered by the customers of an *arbitrary busy-period*) is represented by $D^*(s)$, $D(t)$, d(t), and $\overline{D}$. The additional delay suffered by an *arbitrary customer* is represented by $G^*(s)$, $G(t)$, g(t), and $\overline{G}$.

Now let us follow the derivation from the previous section to derive the LST of the additional delay suffered in the system. The LST of the additional delay suffered by the customers of busy period $j+1$, conditioning that the service time of the first customer of this busy period is z, is recursively expressed as follows:

$$D_{j+1}^*(s \mid z) = \frac{\lambda[D_j^*(\lambda) - D_j^*(s)]}{s - \lambda} + S^*(s \mid z)D_j^*(\lambda) \tag{3.18}$$

Since the system is in equilibrium, the additional delay suffered the customers of the jth busy period is the equilibrium additional delay ($D^*(\lambda)$, so:

$$D_{j+1}^*(s \mid z) = \frac{\lambda[D^*(\lambda) - D^*(s)]}{s - \lambda} + S^*(s \mid z)D^*(\lambda) \tag{3.19}$$

Clearly, the above expression is correct for every j, so it represents the LST of the additional delay suffered by an arbitrary busy period (conditioning on the amount of work started this b.p.). Thus:

$$D^*(s \mid x) = \frac{\lambda[D^*(\lambda) - D^*(s)]}{s - \lambda} + S^*(s \mid x) D^*(\lambda) \tag{3.20}$$

Next, we calculate the distribution of the additional delay suffered by an arbitrary customer. This is done by considering the fraction of customers who are served in busy periods which start with an amount of work equal to x, and by unconditioning equation (3.20). In addition, we substitute $D^*(s)$ and $D^*(\lambda)$ with the corresponding expression taken from (3.5). This calculation yields:

$$G^*(s) = \frac{1}{1+\lambda\overline{S}} \cdot \left[\frac{(s-\lambda)(1-S^*(s))}{s} + \frac{\int_0^\infty b(x) x S^*(s \mid x) dx}{\overline{b}} \right] \tag{3.21}$$

From this equation it is easy to derive the expected additional delay suffered by an arbitrary customer:

$$\overline{G} = \frac{1}{1+\lambda\overline{S}} \cdot \left[\frac{\lambda\overline{S^2}}{2} + \frac{\int_0^\infty \overline{S_{|x}} \cdot b(x) \cdot x \cdot dx}{\overline{b}} \right] \tag{3.22}$$

As mentioned earlier, the expected value of the additional delay can be added to the original expected delay suffered in the M/G/1 system to yield the expected system time spent by an arbitrary customer. Nevertheless, the the LST of the additional delay, cannot be multiplied by the LST of the original delay suffered in the M/G/1 system to yield the LST of the system time in our system. This is true since the original delay is not independent of the additional delay.

In the next section, we use a more sophisticated analysis to derive the LST of the total delay suffered by an arbitrary customer in this type of system.

3.5 The M/G/1 System where the Starter Depends on the Amount of Work Brought by a First Customer: Analysis of the System Time

The approach used in the previous section cannot be used to derive the distribution of the total delay suffered in the system. Even though we derived the LST of the additional delay, $D^*(s)$, and even though the LST of the original delay is known for an M/G/1 system, one cannot directly use these results to compute the LST of the total delay suffered in the system. The reason, as already mentioned, is that in this system, in contrast to the system with an independent starter, the additional delay suffered by an arbitrary customer is *not independent* of the original delay suffered by this customer, so the two transforms cannot be multiplied to give the required LST. Therefore, we have to use a more direct approach to find this LST.

Let us assume that the system is in equilibrium and let $\tilde{x}$ and $\tilde{w}$ be random variables respectively representing the service time and the waiting time of an arbitrary customer in the system *without* starter *(system-A)*. Let $\tilde{d}$ be a random variable representing the additional delay suffered by an arbitrary customer in the system *with* starter *(system-B)*. This additional delay is due to the starter. Let $\tilde{t}$ be a random variable representing the system time (total time) an arbitrary customer suffers in the system *with* starter *(system-B)*. We call these variables the service time ($\tilde{x}$), the original waiting time ($\tilde{w}$), the additional delay ($\tilde{d}$) and the system time ($\tilde{t}$). The probabilistic measures of these variables are the following:

$$B(y) \triangleq Pr[\tilde{x} \leq y], \quad b(y) \triangleq \partial \frac{B(y)}{\partial y}, \quad B^*(s) \triangleq \int_0^\infty e^{-sy} b(y) dy$$

$$W(y) \triangleq Pr[\tilde{w} \leq y], \quad w(y) \triangleq \partial \frac{W(y)}{\partial y}, \quad W^*(s) \triangleq \int_0^\infty e^{-sy} w(y) dy$$

$$D(y) \triangleq Pr[\tilde{d} \leq y], \quad d(y) \triangleq \partial \frac{D(y)}{\partial y}, \quad D^*(s) \triangleq \int_0^\infty e^{-sy} d(y) dy$$

$$T(y) \triangleq Pr[\tilde{t} \leq y], \quad t(y) \triangleq \partial \frac{T(y)}{\partial y}, \quad T^*(s) \triangleq \int_0^\infty e^{-sy} t(y) dy$$

In our analysis we realize that the system time of a given customer depends on two properties of first customers:

1. If the customer is the "first customer" of a busy period or not.

2. The amount of work brought to the system by the "first customer" of the busy period, in which our customer is served.

For the sake of clarity, let us define these properties carefully. Consider the busy and idle periods as observed in system-A (i.e., a system without the starter). Recall that a customer is *a first customer* if it happens to be the first customer of some busy period. A customer which is not the first customer of some busy period is called *non first*. We say that a busy period is of *height* x (or *type* x) if the amount of work brought to the system by the first customer of this busy period is x. We say that a customer *belongs* to a busy period of type x if it is served in a busy period of type x. In figure 3.1 we see that customer j is a first customer, while the customers $j+1,\ldots,j+k$ are non-first. The busy period described in this figure is of type x.

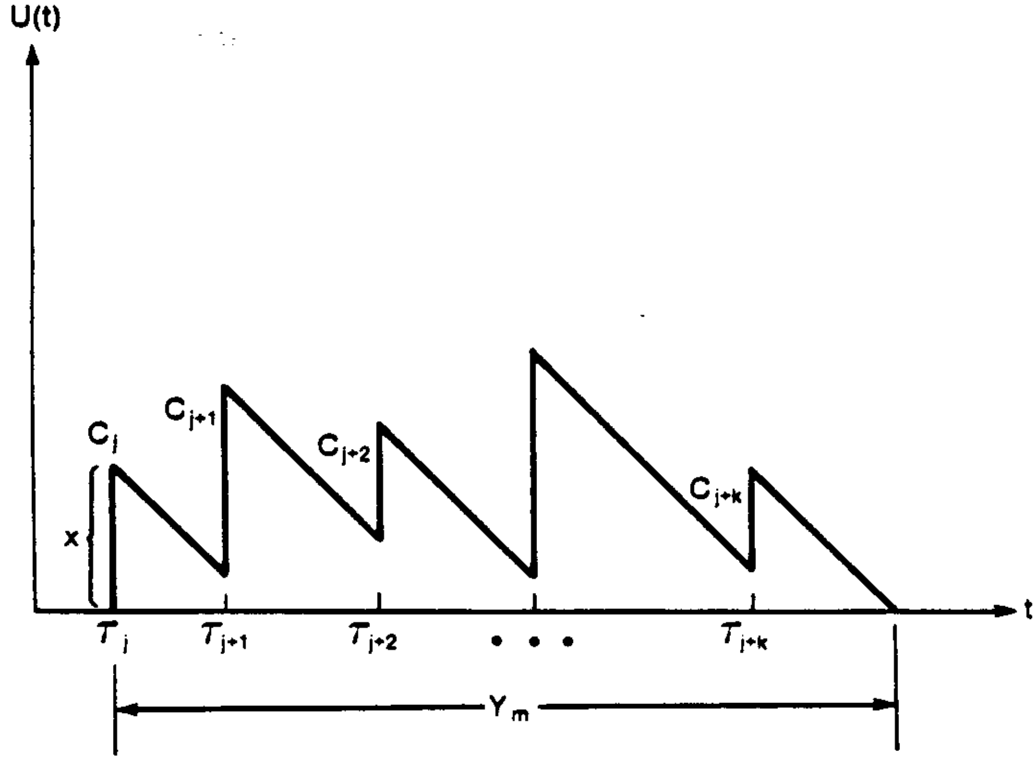

Figure 3.1: The mth busy period, a busy period of type x

Next we are interested in the conditional distributions of the service time, the original waiting time, the additional delay and the system time. These measures are conditioned on the height (type) of the busy period to which the customer belongs and on the customer being first or non-first. First, we denote the service time, waiting time, additional delay and system time of the customers which are non-first, distinguished with the subscript n:

$$B_n(y \mid x) \triangleq Pr[\tilde{x} \leq y \mid \text{the customer is non-first and is served in a b.p. of type } x]$$

$$b_n(y \mid x) \triangleq \frac{\partial B_n(y \mid x)}{\partial y}, \qquad B_n^*(s \mid x) \triangleq \int_0^\infty e^{-sy} b_n(y \mid x) dy$$

$$W_n(y \mid x) \triangleq Pr[\tilde{w} \leq y \mid \text{the customer is non-first and is served in a b.p. of type } x]$$

$$w_n(y \mid x) \triangleq \frac{\partial W_n(y \mid x)}{\partial y}, \qquad W_n^*(s \mid x) \triangleq \int_0^\infty e^{-sy} w_n(y \mid x) dy$$

$D_n(y \mid x) \triangleq Pr[\tilde{d} \le y \mid$ the customer is non-first and is served in a b.p. of type $x]$

$$d_n(y \mid x) \triangleq \frac{\partial D_n(y \mid x)}{\partial y}, \qquad D_n^*(s \mid x) \triangleq \int_0^\infty e^{-sy} d_n(y \mid x) dy$$

$T_n(y \mid x) \triangleq Pr[\tilde{t} \le y \mid$ the customer is non-first and is served in a b.p. of type $x]$

$$t_n(y \mid x) \triangleq \frac{\partial T_n(y \mid x)}{\partial y}, \qquad T_n^*(s \mid x) \triangleq \int_0^\infty e^{-sy} t_n(y \mid x) dy$$

Similar notation is used for first customers, distinguished by subscript f:

$B_f(y \mid x) \triangleq Pr[\tilde{x} \le y \mid$ the customer is first and is served in a b.p. of type $x]$

$$b_f(y \mid x) \triangleq \frac{\partial B_f(y \mid x)}{\partial y}, \qquad B_f^*(s \mid x) \triangleq \int_0^\infty e^{-sy} b_f(y \mid x) dy$$

$W_f(y \mid x) \triangleq Pr[\tilde{w} \le y \mid$ the customer is first and is served in a b.p. of type $x]$

$$w_f(y \mid x) \triangleq \frac{\partial W_f(y \mid x)}{\partial y}, \qquad W_f^*(s \mid x) \triangleq \int_0^\infty e^{-sy} w_f(y \mid x) dy$$

$D_f(y \mid x) \triangleq Pr[\tilde{d} \le y \mid$ the customer is first and is served in a b.p. of type $x]$

$$d_f(y \mid x) \triangleq \frac{\partial D_f(y \mid x)}{\partial y}, \qquad D_f^*(s \mid x) \triangleq \int_0^\infty e^{-sy} d_f(y \mid x) dy$$

$T_f(y \mid x) \triangleq Pr[\tilde{t} \le y \mid$ the customer is first and is served in a b.p. of type $x]$

$$t_f(y \mid x) \triangleq \frac{\partial T_f(y \mid x)}{\partial y}, \qquad T_f^*(s \mid x) \triangleq \int_0^\infty e^{-sy} t_f(y \mid x) dy$$

Now we show that the conditional original waiting time, the conditional service time and the conditional additional delay are independent of each other.

THEOREM 3.1: Given that a customer is served (in system-A) in a busy period of type x, the additional delay it suffers is statistically independent of its original waiting time and of its service time.

Proof: Let us first consider an arbitrary customer, C_i, which is non-first. Let j be the busy period in which C_i is served, and let C_k $(k<i)$ be the first customer served in this busy period. From fact 1, observed in section 3.2, the additional delay suffered by C_i and C_k is the same. Thus, we have to show that conditioning on the type of the busy period (or, actually, conditioning on the service time of C_k) the additional delay suffered by C_k is independent of the original delay suffered by C_i and of the service time of C_i.

It is clear that the original delay suffered by C_i is a function only of the service times and interarrival times that "belong" to busy period j. Namely, the sequence $c_k, c_{k+1}, \cdots, c_{i-1}$ and the sequence $t_{k+1}, t_{k+2}, \cdots, t_i$. Let us call this group of variables, the original delay group. On the other hand, the additional delay suffered by C_k is only a function of the system behavior prior to τ_k, the starting time of busy period j. More specifically, this is a function only of the sequence $t_2, t_3, \cdots t_k$ the sequence $c_1, c_2, \cdots c_{k-1}$ and the sequence $S_1, S_2, \cdots, S_j$. More carefully, we recall that S_j is a function of c_k (the cold start depends on the service time of the first customer in the busy period). Thus, we have to add to this set the variable c_k. Let us call this set (including c_k) the additional delay group. In addition, we must mention that $S_1, \cdots S_{j-1}$ also depend on the service times observed in the system; however, all these variables belong to the "past" and are contained in the set $c_1, c_2, \cdots c_{k-1}$.

Now, we note that the additional delay group and the original delay group are mutually exclusive, excluding c_k that belongs to both groups. Due to the assumptions on the arrival process (independence of interarrival times and service times) it is clear that the statistical dependence between the groups is only due to the fact that c_k belongs to both groups. Therefore, we conclude that if we condition on the value of c_k (which is identical to the busy period type) the additional delay suffered by C_k is independent of the original delay suffered by C_i.

It is now easy to prove that the service time of C_i is independent of the additional delay it suffers. This is true since this variable does not belong to the additional delay group.

Now, let us consider a customer C_i which is a first customer. First, the original delay suffered by this first customer is constant (0), so it is independent of the additional delay it suffers. Second, considering the customer's service time, c_i, we realize that it is contained in the original delay group. However, since we condition on the value of c_i, the conditional service time is independent of the conditional additional delay. ■

In addition to the above theorem, it is well known that the waiting time of an arbitrary customer in an M/G/1 system is independent of its service time. Therefore, the original waiting time (in our system) is independent of the service time. All this leads to the following conclusion:

COROLLARY: The conditional system time consists of the sum of three *independent* random variables: the conditional original waiting time, the conditional service time and the conditional additional delay.

From this corollary it is now easy to calculate the LST of the conditional system time. This is simply the multiplication of the proper Laplace transforms:

$$T_f^*(s \mid x) = B_f^*(s \mid x) \cdot W_f^*(s \mid x) \cdot D_f^*(s \mid x) \tag{3.23}$$

$$T_n^*(s \mid x) = B_n^*(s \mid x) \cdot W_n^*(s \mid x) \cdot D_n^*(s \mid x) \tag{3.24}$$

We are now ready to start calculating the conditional system time. This we do by using equations (3.23) and (3.24) and by calculating, one by one, the proper terms appearing in these equations.

The Delay Suffered by First Customers

In the following we derive the LST of the conditional service time ($B_f^*(s \mid x)$) and the LST of the conditional original waiting time ($W_f^*(s \mid x)$) suffered by first customers. The LST of the conditional additional delay ($D_f^*(s \mid x)$) will be derived later.

First, it is clear that since we condition on the fact that the first customer is served in a busy period of type x then its service time is exactly x, so the LST of its conditional service time is:

$$B_f^*(s \mid x) = e^{-sx} \tag{3.25}$$

Second, the original waiting time of first-customers is always zero, so:

$$W_f^*(s \mid x) = 1 \tag{3.26}$$

The Delay Suffered by Non-First Customers

In the following we derive the LST of the conditional service time ($B_n^*(s \mid x)$) and the LST of the conditional original waiting time ($W_n^*(s \mid x)$) suffered by non-first customers.

First, it is clear that the service time of a non-first customer is independent of the type of the busy period to which this customer belongs. Thus, its service time is just the service time of an arbitrary customer in a regular M/G/1 system:

$$B_n^*(s \mid x) = B^*(s) \tag{3.27}$$

The computation of the LST of the original waiting time of non-first customers is more complicated. This is true since the waiting time that a non-first customer suffers in system-A, strongly depends on the type of busy period to which this customer belongs.

To analyze this LST, we follow the method used in [Klei75] to derive the waiting time in an M/G/1 system by analyzing the behavior of the busy period; we depart from this M/G/1 derivation when we uncondition on x, the height of the busy period.

Let us consider a busy period of type x, as observed in system-A. Let us examine the unfinished work, $U(t)$, as observed in the system during this busy period. To analyze the waiting time we decompose this busy period into a sequence of intervals whose length are dependent random variables as follows. Consider this busy period as shown in figure 3.2 (borrowed from [Klei75] and modified properly). Here we see that customer C_1 initiates the busy period upon his arrival at time τ_1. The first interval we consider is his service time $c_1 = x$, which we denote by X_0; during his interval more customers arrive (in this case C_2 and C_3). All those customers who arrive during X_0 are served in the next interval, whose duration is X_1 and which equals the sum of the service times of all arrivals during X_0 (in this case C_2 and C_3). At the expiration of X_1 we then create a new interval of duration X_2 in which all customers arriving during X_1 are served, and so on. Thus, X_i is the length of time required to service all those customers who arrive during the previous interval whose duration is X_{i-1}. Now, let us denote by n_i ($i=0,1,2,...$) the number of customer arrivals during the interval X_i. Using this notation it is clear that n_i customers are served during the interval X_{i+1}. To allow for an infinite number of such intervals, we define $X_i = 0$ for those intervals that fall beyond the termination of this busy period.

Let us now define $X_i(y)$ to be the PDF for X_i, and $X_i^*(s)$ the LST of the corresponding pdf, that is,

$$X_i(y) \triangleq Pr[X_i \leq y] \quad , \quad X_i^*(s) \triangleq \int_0^\infty e^{-sy} dX_i(y) = E[e^{-sX_i}]$$

In a similar manner, we define the following measures for, conditioning on the fact that the busy period to which this interval belongs is of type x:

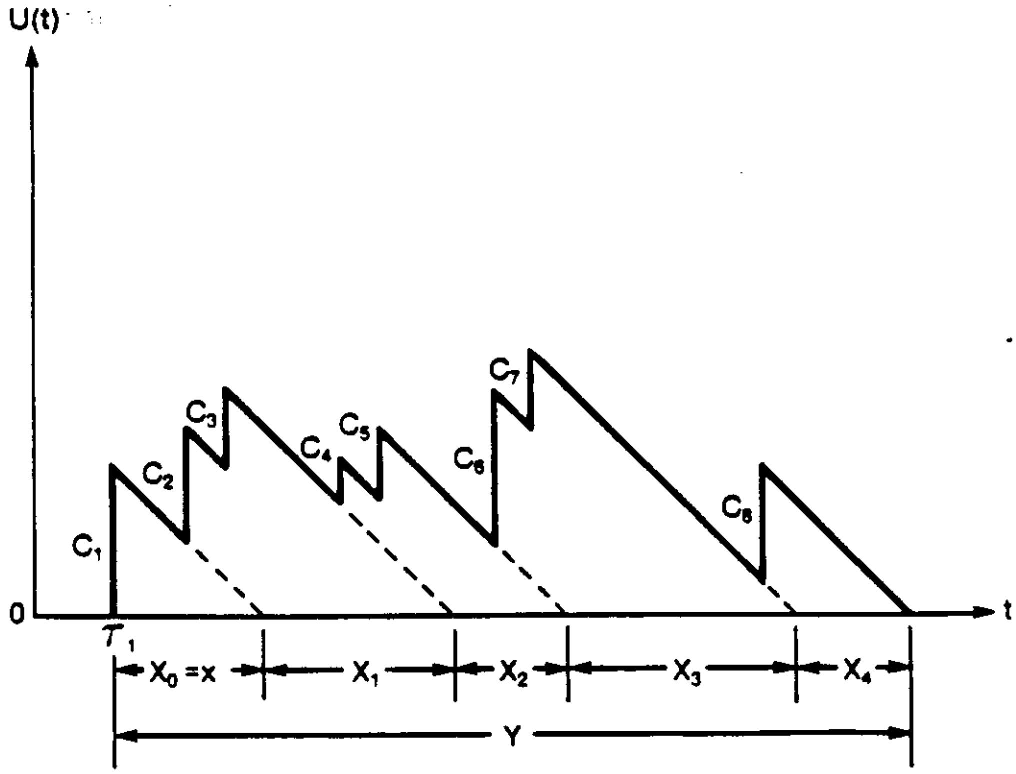

Figure 3.2: The sub-busy periods of a busy period of type x

$$X_i(y \mid x) \triangleq Pr[X_i \le y \mid X_i \text{ falls in b.p. of type } x] \quad , \qquad X_i^*(s \mid x) \triangleq \int_0^\infty e^{-sy}\, dX_i(y \mid x)$$

We now condition our calculation on the event that a new tagged customer arrives while the busy period is in its ith duration (i.e, arrives during the interval X_i); let $\tilde{w}$ denote the waiting time of this customer. It is clear that the waiting time of our tagged customer equals the remaining time of the ith interval plus the service time of all the customers* who arrived before he did during the ith interval. Let us define Y_i to be the remaining time of the ith interval (that is, the time from the arrival of our tagged customer until the end of the ith interval), and let N_i be the number of arrivals during the ith interval but prior to the arrival of our tagged customer (that is, in the interval $X_i - Y_i$). Now, we are interested in calculating $E[e^{-s\tilde{w}} \mid i, X_0 = x]$, and this we do by conditioning on X_i, Y_i and N_i. This is simply the following:

$$E[e^{-s\tilde{w}} \mid i, X_i = y, Y_i = y', N_i = n, X_0 = x] = e^{-sy'}\left[B^*(s)\right]^n \tag{3.28}$$

Now, since we assume that n customers have arrived during an interval of duration $y - y'$, we uncondition on N_i as follows:

$$E[e^{-s\tilde{w}} \mid i, X_i = y, Y_i = y', X_0 = x] = e^{-sy'} \sum_{n=0}^{\infty} \frac{[\lambda(y-y')]^n}{n!} e^{-\lambda(y-y')}\left[B^*(s)\right]^n$$

$$= e^{-sy' - \lambda(y-y') + \lambda(y-y')B^*(s)} \tag{3.29}$$

Since Y_i is defined to be the remaining time of the ith interval, it is distributed as the residual life of X_i. Therefore, as shown in [Klei75] the joint density for the conditional X_i and the conditional Y_i is given by:

$$Pr[y < X_i \le y + dy,\ y' < Y_i \le y' + dy' \mid i, X_0 = x] = \frac{dX_i(y \mid x)dy'}{E[X_i \mid x]} \qquad \text{for } 0 \le y' \le y \le \infty$$

Thus, we can uncondition on X_i and Y_i in equation (3.29) to obtain:

$$E[e^{-s\tilde{w}} \mid i, X_0 = x] = \int_{y=0}^{\infty} \int_{y'=0}^{y} \left(e^{-[s-\lambda+\lambda B^*(s)]y'} \cdot e^{-[\lambda-\lambda B^*(s)]y}\right) \cdot \frac{dX_i(y \mid x)dy'}{E[X_i \mid X_0 = x]}$$

$$= \int_{y=0}^{\infty} \frac{\left[e^{-sy} - e^{-[\lambda-\lambda B^*(s)]y}\right]}{[-s+\lambda-\lambda B^*(s)] \cdot E[X_i \mid X_0 = x]} dX_i(y \mid x) \tag{3.30}$$

which is identical to:

$$E[e^{-s\tilde{w}} \mid i, X_0 = x] = \frac{X_i^*(s \mid x) - X_i^*(\lambda - \lambda B^*(s) \mid x)}{[-s+\lambda-\lambda B^*(s)] \cdot E[X_i \mid X_0 = x]} \tag{3.31}$$

Now, $X_{i+1}^*(s \mid x)$ can easily be recursively expressed in terms of $X_i^*(s \mid x)$ (see the derivation of equation 5.161 in [Klei75]) as follows:

*We assume that the service policy is first come first served (FCFS)

$$X^*_{i+1}(s \mid x) = X^*_i\left(\lambda-\lambda B^*(s) \mid x\right) \tag{3.32}$$

Thus, using equation (3.32) in equation (3.31) we get:

$$E[e^{-s\tilde{w}} \mid i, X_0=x] = \frac{X^*_{i+1}(s \mid x)-X^*_i(s \mid x)}{[s-\lambda+\lambda B^*(s)]\cdot E[X_i \mid X_0=x]} \tag{3.33}$$

Next we note that the probability of arriving into the ith interval is proportional to its length, so:

$$Pr[\text{arriving in the } i\text{th interval} \mid \text{the b p. is of type } x] = \frac{E[X_i \mid x]}{E[Y \mid X_0=x]}$$

Thus, we can remove from equation (3.33) the condition that our customer arrived in the ith interval (still conditioning on the fact that it arrived *during* a busy period):

$$E[e^{-s\tilde{w}} \mid \textit{customer arrives during a b.p.}, X_0=x] = \frac{\sum_{i=0}^{\infty} X^*_{i+1}(s \mid x)-X^*_i(s \mid x)}{[s-\lambda+\lambda B^*(s)]\cdot E[Y \mid X_0=x]} \tag{3.34}$$

Now, the summation from this equation collapses into $X^*_\infty(s \mid x)-X^*_0(s \mid x)$, which is clearly identical to $1-e^{-sx}$ (recall that $X_\infty=0$), so:

$$E[e^{-s\tilde{w}} \mid \textit{customer arrives during a b.p.}, X_0=x] = \frac{1-e^{-sx}}{[s-\lambda+\lambda B^*(s)]\cdot E[Y \mid X_0=x]} \tag{3.35}$$

Now, let us calculate $E[Y \mid X_0=x]$. First it is known that the expected length of an arbitrary busy period is given by

$$E(Y) = \frac{\bar{x}}{1-\rho} \tag{3.36}$$

where $\bar{x}$ is the expected service time and ρ is the system utilization.

Second, we note that the distribution of the amount of work brought to the system by the first customer of a busy period is just the distribution of the service time, $b(x)$. Therefore, we get:

$$E(Y) = \int_0^\infty E[Y \mid x]b(x)dx$$

so:

$$\int_0^\infty E[Y \mid x]b(x)dx = \frac{\bar{x}}{1-\rho} \tag{3.37}$$

Now, in our system, as opposed to a regular M/G/1 system, we condition on the amount of work

(x) brought by its first customer. Thus, the expected length of a b.p. is proportional to x, so:

$$E[Y \mid X_0=x] = c \cdot x$$

where c is some constant. Thus, we obtain:

$$\int_0^{\infty} E[Y \mid x] b(x) dx = \int_0^{\infty} c \cdot x \cdot b(x) dx = c \cdot \overline{x} \tag{3.38}$$

Using (3.37) and (3.38) we get that $c=\frac{1}{1-\rho}$ and the desired result is:

$$E[Y \mid X_0=x] = \frac{x}{1-\rho} \tag{3.39}$$

Finally, we can now plug (3.39) into (3.35) to yield the conditional LST of the original waiting time of non-first customers:

$$W_n^*(s \mid x) = E[e^{-s\tilde{w}} \mid \text{customer arrives during a b.p. of type } x] = \frac{(1-e^{-sx})\cdot(1-\rho)}{(s-\lambda+\lambda B^*(s))\cdot x} \tag{3.40}$$

The Distribution of the Conditional Additional Delay

In the following we derive the expressions for the LST of the additional delay suffered by first customers and non-first customers as proposed.

From the analysis done in section 3.4 it is clear that the additional delay suffered by the customers of a busy period of type x is identical for all such customers, independent of the first/non-first property. From that analysis we get:

$$D_n^*(s \mid x) = D_f^*(s \mid x) = D^*(s \mid x) = \frac{\lambda[D^*(\lambda)-D^*(s)]}{s-\lambda}+S^*(s \mid x)\cdot D^*(\lambda) \tag{3.41}$$

The LST of the Total Delay Suffered in the System

Having calculated the LST of the conditional service time, the conditional original waiting time and the conditional additional delay, we are now ready to calculate the LST of the total system time.

The LST of the total delay suffered by first customers which belong to a busy period of type x is given by:

$$T_f^*(s \mid x) = 1\cdot e^{-sx}\cdot\left[\frac{\lambda\cdot[D^*(\lambda)-D^*(s)]}{s-\lambda} + S^*(s \mid x) D^*(\lambda)\right] \tag{3.42}$$

where equation (3.26) gives the factor 1, equation (3.25) gives the factor e^{-sx} and equation (3.41) gives the term in brackets.

The LST of the total delay suffered by a non-first customer who belongs to a busy period of type x is given by:

$$T_n^*(s \mid x) = \frac{(1-e^{-sx})(1-\rho)}{[s-\lambda+\lambda B^*(s)]x} \cdot B^*(s) \cdot \left[\frac{\lambda \cdot [D^*(\lambda)-D^*(s)]}{s-\lambda} + S^*(s \mid x) D^*(\lambda) \right] \qquad (3.43)$$

where the first term, the second term and the third term are taken from equations (3.40), (3.27) and (3.41), respectively.

Next, we calculate the fraction of customers which belong to each type.

Considering the "first" customers, we realize that these are actually customers who arrive to system-A while the system is empty. Therefore, the fraction of customers that are first customers is exactly $1-\rho$, where ρ is the utilization factor of system-A (recall that $\rho=\lambda\bar{x}$). Now, it is clear that the first customers who "fall" into a busy period of type x are exactly those first customers whose service time is x. Thus, the fraction of first customers who "fall" into a busy period of type x is given by $b(x)$. Therefore, we conclude that the fraction of customers who are "first" customers and belong to a busy period of type x is given by:

$$P_{f,x} \triangleq Pr[\text{ a customer is a first customer and belongs to a b.p. of type } x\ ] = b(x)\cdot(1-\rho) \qquad (3.44)$$

Considering the non-first customers, it is now clear that these are customers who arrive to system-A while it is busy, so the fraction of customers who are non-first is exactly ρ. To find the fraction of non-first customers which are served in a busy period of type x, we note that the expected length of a busy period is proportional to the amount of work brought by the customer starting the busy period. Thus, the expected number of customers arriving to the system during a busy period of type x, is proportional to $b(x)\cdot x$ Therefore, the fraction of non-first customers who belong to a busy period of type x is given by:

$$Pr[\text{ a customer belongs to a b.p. of type } x \mid \text{non-first customer}\ ] = \frac{b(x)\cdot x}{\int_0^\infty b(x)\cdot x dx} = b(x)\cdot x\cdot\lambda$$

And the fraction of customers who are non-first and served in a busy period of type x is given by:

$$P_{n,x} \triangleq Pr[\textit{ a customer is a non-first customer} \text{ and } \textit{belongs to a b.p. of type-}x\ ] =$$

$$= \frac{b(x)\cdot x}{\int_0^\infty b(x)\cdot x dx}\cdot\rho = \frac{b(x)\cdot x\cdot\rho}{\bar{x}} \qquad (3.45)$$

Having calculated the conditional LST of the system time and the fraction of customers belong to each type, it is now straightforward to calculate the LST of the system time for an arbitrary customer.

$$T^*(s) = \int_0^\infty P_{l,x} T_l^*(s \mid x)dx + \int_0^\infty P_{n,x} T_n^*(s \mid x)dx \tag{3.46}$$

Substituting (3.42), (3.43), (3.44) and (3.45) into (3.46) yields:

$$T^*(s) = \int_0^\infty (1-\rho)b(x)e^{-sx}\cdot\left[\frac{\lambda\cdot[D^*(\lambda)-D^*(s)]}{s-\lambda} + S^*(s \mid x)D^*(\lambda)\right]\cdot dx$$

$$+ \int_0^\infty \lambda\cdot b(x)x\cdot\frac{(1-e^{-sx})(1-\rho)}{(s-\lambda+\lambda B^*(s))\cdot x}\cdot B^*(s)\cdot\left[\frac{\lambda\cdot[D^*(\lambda)-D^*(s)]}{s-\lambda} + S^*(s \mid x)D^*(\lambda)\right]\cdot dx$$

Using the following equalities:

$$\int_0^\infty b(x)e^{-sx}dx = B^*(s) \ , \quad \int_0^\infty b(x)dx = 1$$

and rearranging the expression for $T^*(s)$, we get:

$$T^*(s) = (1-\rho)\cdot\frac{\lambda\cdot[D^*(\lambda)-D^*(s)]}{s-\lambda}\cdot B^*(s) + \frac{(1-\rho)}{s-\lambda+\lambda B^*(s)}\cdot\lambda B^*(s)\cdot\frac{\lambda\cdot[D^*(\lambda)-D^*(s)]}{s-\lambda}\cdot(1-B^*(s))$$

$$+ \frac{(1-\rho)}{s-\lambda+\lambda B^*(s)}\cdot\lambda B^*(s)D^*(\lambda)S^*(s) + \left[(1-\rho)D^*(\lambda) - \frac{1-\rho}{s-\lambda+\lambda B^*(s)}\cdot\lambda B^*(s)D^*(\lambda)\right]\cdot\left[\int_0^\infty b(x)e^{-sx}S^*(s \mid x)dx\right]$$

Factoring the first and the second terms of the above expression and combining them together, and factoring the last term gives:

$$T^*(s) = (1-\rho)\cdot\frac{\lambda\cdot[D^*(\lambda)-D^*(s)]}{s-\lambda}\cdot B^*(s)\cdot\frac{s}{s-\lambda+\lambda B^*(s)} + \frac{(1-\rho)}{s-\lambda+\lambda B^*(s)}\cdot\lambda B^*(s)D^*(\lambda)S^*(s)$$

$$+ (1-\rho)D^*(\lambda)\cdot\frac{s-\lambda}{s-\lambda+\lambda B^*(s)}\cdot\left[\int_0^\infty b(x)e^{-sx}S^*(s \mid x)dx\right] \tag{3.47}$$

Now, from (3.4) it is clear that:

$$D^*(\lambda) = \frac{1}{1+\lambda\overline{S}}$$

thus, we get:

$$\frac{D^*(\lambda)-D^*(s)}{s-\lambda} = \frac{D^*(\lambda)[1-S^*(s)]}{s}$$

Substituting this relation into (3.47) yields:

$$T^*(s) = \frac{(1-\rho)}{s-\lambda+\lambda B^*(s)}\cdot\lambda B^*(s)D^*(\lambda)\cdot[1-S^*(s)] + \frac{(1-\rho)}{s-\lambda+\lambda B^*(s)}\cdot\lambda B^*(s)D^*(\lambda)S^*(s)$$

$$+ (1-\rho)D^*(\lambda)\cdot\frac{s-\lambda}{s-\lambda+\lambda B^*(s)}\cdot\left[\int_0^\infty b(x)e^{-sx}S^*(s\mid x)dx\right]$$

Rearranging this expression and substituting $D^*(\lambda)$ from (3.4), finally gives the desired result, the LST of the system time:

$$T^*(s) = \frac{(1-\rho)}{[s-\lambda+\lambda B^*(s)]\cdot[1+\lambda\overline{S}]}\cdot\left[\lambda B^*(s)+(s-\lambda)\cdot\int_0^\infty b(x)e^{-sx}S^*(s\mid x)dx\right] \qquad (3.48)$$

Now, if one is interested in calculating the LST of the "additional delay" suffered in the system, (3.48) can be rewritten as following:

$$T^*(s) = \left\{\frac{(1-\rho)sB^*(s)}{s-\lambda+\lambda B^*(s)}\right\}\cdot\left[\frac{1}{1+\lambda\overline{S}}\cdot\left(\lambda+\frac{(s-\lambda)\cdot\int_0^\infty b(x)e^{-sx}S^*(s\mid x)dx}{B^*(s)}\right)\right] \qquad (3.49)$$

The first term in (3.49) is the well known expression for the system time in the M/G/1 *without* starter. Thus, the second term of (3.49) could be *considered* as the additional delay suffered in the system with non-independent starter, and we can consider the delay suffered in this system as consisting of the sum of two independent random variables: 1) the delay in the M/G/1 without starter, represented by the first term. 2) The additional delay suffered due to the starter and represented by the second term.

The reader should note the difference between the property mentioned above for the system with a non-independent starter to the equivalent property observed for the system with an independent starter studied in chapter 2. In chapter 2 we could show that for the system with an independent starter the total delay *is* distributed as the sum of two random variables (the original delay and the additional delay); moreover, we have shown how to directly calculate the additional delay in that system. In contrast, in the analysis of the non-independent starter system, done above, we can only *consider the delay expression as consisting* of the sum of two random variables.

3.6 A System where the Starter Depends on the Length of the Idle Period

In this section we are interested in analyzing a system where the start-up delay depends on the amount of time the system was idle before the arrival of a "first customer". Such a system, where the underlying system is M/G/1, has been analyzed by Welch [Welc64]. In the following we analyze this system using our approach. Like Welch we do the analysis for an M/G/1 system with single arrivals. In addition we emphasize that the same approach can easily be used to analyze discrete systems and systems with bulk arrivals. This is an immediate result of the method of analyzing the additional delay instead of the total delay.

The basic assumptions required for this section are:

1. The interarrival times possess the memoryless property. Thus in a continuous system the interarrival times are exponentially distributed and in the discrete system they are geometrically distributed.

2. For the analysis of the additional delay the arrival may consist either of bulks or of single customers.

3. The length of a "cold start" depends on the length of the idle period preceding it. For clarity it should be emphasized that this dependence is on the length of the idle period as observed in *system-B*. This emphasis is important for having a realistic model.

Under these assumptions it is easy to see that the following holds:

1. The cold start (if any) suffered by the jth busy period (according to system-A) is *independent* of any property of this busy period.

2. The length of an idle period, as observed in system-B is exponentially distributed with parameter λ.

3. Let us assume that the jth busy period (according to system-A) suffered a cold start in system-B. Let t_0 be the moment when system-B becomes idle prior to the beginning of the jth busy period. Then the cold start suffered by the jth busy period is *independent* of the history of system-B prior to t_0.

Let us now calculate the additional delay suffered in an M/G/1 system. The notation to be used is very similar to the notation used in section 3.4. In fact, we use exactly the same notation but modify its meaning. The modification is that here the conditioning parameter (x) represents the length of the preceding idle period while in section 3.4 it represented the amount of work brought to the system by the first customer. Thus, for example, we denote by $S_j^*(s\,|x)$ the LST of the cold start (if any) suffered by the jth busy period (according to the realization in system-A) conditioning on the fact that the length of the preceding idle period (according to system-B) is x.

From the observations made above it is clear that in order to compute the LST of the additional delay suffered by an arbitrary busy period, one need merely to calculate the LST of an arbitrary cold start and then plug it into equation (3.5). From the second observation, calculating the LST of the cold start suffered by an arbitrary busy period is an easy task:

$$S^*(s) = \int_0^\infty \lambda e^{-\lambda x} S^*(s \mid x) dx \qquad (3.50)$$

Similarly one can easily calculate the expected value of the cold start distribution:

$$\overline{S} = \int_0^\infty \lambda e^{-\lambda x} \overline{S_{|x}} dx \qquad (3.51)$$

And, as observed above, the LST of the additional delay is expressed in the terms of $S^*(s)$ and $\overline{S}$:

$$D^*(s) = \frac{1}{1+\lambda\overline{S}} \cdot \left[\frac{\lambda + S^*(s)(s-\lambda)}{s} \right] \qquad (3.52)$$

which is what we are looking for.

It is easy to see that the above results hold for M/G/1 systems where the arrivals consist of customer bulks as well as for an M/G/1 system where the arrivals consist of single customers. In addition, we realize that the same approach can be used to analyze discrete systems where the interarrival times are geometrically distributed. As observed for the M/G/1 system, the results for the discrete system will hold for bulk arrivals as well as for single customer arrivals. While the basic result (concerning the M/G/1 system with single customer arrivals) has been reported by Welch, the extensions for discrete systems and bulk arrival systems have not been observed previously.

3.6.1 A Note on Mixed Systems

In the first sections of this paper we considered systems where the start-up delay depends on the amount of work it finds in the system. In the beginning of section 3.6 we dealt with a system where the start-up delay depends on the amount of time the system were idle. A natural extension of this work is to consider mixed systems. *A mixed system* is a system where the start-up delay depends both on the amount of work it finds in the system and on the length of the preceding idle period. Using the results reported in the previous sections it is quite easy to analyze mixed systems.

For simplicity let us consider an M/G/1 system and use the notation used in sections 4 and 5. Thus, $S^*(s)$ denotes the LST of an arbitrary cold start and $S^*(s \mid x)$ denotes the LST of a cold start conditioning that the amount of work brought to the system by the first customer of the busy period (corresponding to the cold start) is x. In addition to the notation used in section 3.4 let us denote by $S^*(s \mid x,y)$ the LST of the cold start conditioning on the fact that the length of the previous idle period is y and the amount of work brought by the first customer of the busy period is x. Similarly, we denote by $\overline{S_{|x,y}}$ the expected value of the cold start conditioning on the same two parameters.

To analyze the system we note that the service time of the first customer of some busy period (according to system-A) is independent of the length of the preceding idle period (according to system-B). Thus, the same approach used in equation (3.50) can be used to uncondition $S^*(s \mid x,y)$. This is simply:

$$S^*(s \mid x) = \int_0^\infty \lambda e^{-\lambda y} S^*(s \mid x,y) dy \tag{3.53}$$

and similarly, one can find $\overline{S_{|x}}$:

$$\overline{S_{|x}} = \int_0^\infty \lambda e^{-\lambda y} \overline{S_{|x,y}} dy \tag{3.54}$$

Now we note that $S^*(s \mid x)$ and $\overline{S_{|x}}$ have the same meaning as in sections 3.4 and 4.5. Thus, to obtain the LST of the additional delay suffered by an arbitrary customer in the mixed system one need merely to use the expression derived in equation (3.53) and then substitute it into equation (3.21). Similarly, to obtain the expected value of the additional delay suffered in the system one need to substitute the expression from equation (3.54) into equation (3.22). In a similar manner, it is easy to derive the LST of the total delay suffered in the system. This can be done by substituting equation (3.53) into equation (3.49).

3.7 Extensions and Generalizations of the Results

In this section we review the results reported in the previous sections and describe how they can be used in the analysis of other systems.

In section 3.3 we analyzed the LST and the expected value of the additional delay suffered in a bulk-arrival discrete time system where the starter depends on the size of the first bulk arriving to an empty system. In a very similar manner one can find the additional delay in a discrete system with *simple arrivals* where the starter depends on the amount of work brought to the system by the *first customer* of a busy period. The z-transform and the expected value of the additional delay suffered by an arbitrary customer are then given by the following equations:

$$G(z) = \frac{1}{1+(1-z)\overline{S}} \cdot \left[\frac{(1-z)(1-S(z))}{1-z} + \frac{\sum_{k=1}^{\infty} S(z \mid k) b_k k}{\overline{b}} \right] \tag{3.55}$$

$$\overline{G} = \frac{1}{1+(1-z)\overline{S}} \cdot \left[\frac{(\overline{S^2}-\overline{S})(1-z)}{2} + \frac{\sum_{k=1}^{\infty} \overline{S|_k} \cdot b_k k}{\sum_{k=1}^{\infty} b_k k} \right] \tag{3.56}$$

which are equivalent to equations (3.15) and (3.17) respectively. The difference is in the interpretation of the expressions, where here (in contrast to the expressions derived in section 3.3) the subscript k appearing in the equations represents the amount of work (in terms of time units) brought to the system by the first customer.

Using the same approach it is also possible to find the additional delay in discrete system with bulk arrivals where the starter depends on the amount of work brought by the *first bulk* arriving to an empty system. The expressions for the additional delay suffered in this system are, again given by equations (3.55) and (3.56) where the interpretation of k and b_k changes properly.

It should be noted that the type of analysis done in section 3.3 is good for all these systems, mainly because it calculates the additional delay suffered in the system (not the total delay!). Since the additional delay depends only on the interarrival times (not on the service times and not on the bulk sizes) it is natural that this can be extended to all these systems.

The same observation made above with respect to discrete systems, can be made when we consider continuous systems. Thus, all the systems described above can be analyzed when the underlying model is a continuous time system. For example the following expressions give the LST and the expected value of the additional delay in a continuous time system with bulk arrivals where the start-up delay depends on the number of customers arriving in a first bulk:

$$G^*(s) = \frac{1}{1+\lambda\overline{S}} \cdot \left[\frac{(s-\lambda)(1-S^*(s))}{s} + \frac{\sum_{k=1}^{\infty} b_k \cdot k S^*(s\ |k)}{\sum_{k=1}^{\infty} b_k \cdot k} \right] \tag{3.57}$$

$$\overline{G} = \frac{1}{1+\lambda\overline{S}} \cdot \left[\frac{\lambda\overline{S^2}}{2} + \frac{\sum_{k=1}^{\infty} \overline{S|_k} \cdot b_k \cdot k}{\sum_{k=1}^{\infty} b_k \cdot k} \right] \tag{3.58}$$

In these expression k denotes the number of customers arriving in a first bulk and b_k denotes the probability that a bulk consists of k customers.

The only part of this work that cannot easily be modified is the analysis of the LST of the *total delay* done in section 3.5. We believe that a similar approach, to the one taken in section 3.5, can be used to analyze the equivalent discrete system. However, this analysis is not trivial. Another non-trivial task is to analyze the LST of the total delay suffered in systems with bulk arrivals.

The extension applied to the work done in sections 3.3 and 3.4, can easily be applied to the systems reported in section 3.6. These are the systems where the starter depends on the length of the previous idle period and the mixed systems. Thus, if one is interested in analyzing these types of systems, it is relatively easy to modify the equations reported in section 3.6 in order to achieve the LST (and the expected value) of the additional delay suffered in the specific system. This analysis can be applied to discrete systems as well as to continuous systems, and to systems with simple arrivals or bulk arrivals.

It turns out that for most types of systems it is relatively easy to derive the LST and the expected value of the additional delay. On the other hand, deriving the LST of the total delay suffered in the system is not so easy, and may be difficult for some of the systems. It is noticed that the system with starter will be difficult to analyze if the equivalent system without starter is difficult to analyze. Nevertheless, it should be noted that the expected value of the additional delay suffered in the system with starter may have its own importance even in the cases where it is difficult to calculate the total delay. The reasons for this are the following: 1) In many cases the measure of interest is the expected value of the delay suffered in the system. In these cases it is valid to calculate the expected value of the original delay (suffered in the system without starter) and to add it to the expected value of the additional delay (suffered in the system with starter). This summation is valid although the two variables are not independent of each other. 2) In some cases it is required to compare two types of starters on the same "original system". In this case it is very important to find both the expected value and the LST of the additional delay even if it is difficult to calculate the same measures for the total delay.

3.8 Summary

We studied queueing systems in which a non-independent start-up delay is incurred in the beginning of each busy period. The particular systems analyzed in this chapter are those where the start-up delay depends on either or both of the following parameters:

1. The amount of work brought to the system by the first customer(s) of a busy period, or, similarly, the number of customers arriving in the first bulk of a busy period.

2. The length of the previous idle period.

The analysis was done for both discrete-time and continuous-time models where the interarrival times posses the memoryless property.

CHAPTER 4
The Analysis of Random Polling Systems

In this chapter we analyze the behavior of *random polling systems*. The polling systems considered consist of N stations, each of them equipped with an infinite buffer, and of a single server who serves them in some order. In contrast to previously studied polling systems where the order of service used by the server is *periodic* (and usually *cyclic*), in the systems considered here the next station to be served after station i is determined by *probabilistic means*. More specifically, according to the model considered in this chapter, after serving station i the server will poll station j ($j=1,2,\cdots,N$) with probability p_j.

The main goal of this chapter is to derive the expected delay in the random polling system. This result is later used in chapter 5 in the analysis of exhaustive slotted ALOHA schemes. In addition we analyze the cycle time and the number of customers found in the system.

4.1 Introduction and Previous Work

The queueing behavior of *polling systems* has been extensively investigated in the last twenty years. The "traditional" polling scheme that appears in the literature, is a method by which a single server serves N stations, each of them generates its own stream of work requests (or customers) and each of them equipped with a queue to store its requests. According to this scheme (called the *cyclic polling scheme*), the N stations are served in a *cyclic order* in which the station served after station i is station $i+1$ (modulo N).

In contrast to previous studies that dealt with *cyclic polling* schemes, our aim, in this chapter, is to study the *random polling* scheme. In a random polling scheme, the station polled after station i is not determined ahead of time. Rather, this station is determined at operation time according to some random criteria. According to the specific scheme we investigate in this chapter, after servicing station i, the server will poll station j ($j=1, 2, \cdots, N$) with probability p_j.

Naturally, the previous studies were motivated by the wish to model a time shared system: the most common computer system of the last twenty years. The time sharing system consists of a *central* processor (controller) which serves many users, by a way of polling them in cyclic order.

In contrast to the past studies, this chapter was motivated by the wish to model a *distributed* system. Unlike the central processor, the distributed system can not always take a decision in a deterministic way. In particular, visiting N stations in fixed order (which is equivalent to moving the control from one station to the next one) may not be a natural process for these systems.

Motivated by the recent developments of distributed systems, we believe that the random polling scheme is a natural model for distributed systems where the control moves from station to station according to some random criteria. As an example (that, as a matter of fact, motivated this research) one can think about a shared channel communication network, where the decision of "who will talk next" is done in distributed manner, and is based on some randomly behaving algorithms, rather than being based on a fixed order. Such a system is analyzed in chapter 5 of this dissertation by using the results derived in this chapter.

The aim of this chapter, is to study the random polling scheme, emphasizing the analysis of the delay suffered in the system. The model we use is a discrete time model and the extension of the results to a continuous time model can be done in a similar way. As done in the analysis of cyclic polling systems, we allow the server to have a random-length *switch over period* between the service of one station to the service of the next station. The length of a switch-over period, in our model, is associated with the station served prior to the switch over period.

For this model, under the assumption of fully symmetric system, we are able to derive a closed form expression of the expected delay in all three types of service policies (the three policies are described below). For a non symmetric system, we form a set of N^2 linear equations that can be solved by numerical methods. A solution of this set easily gives the expected delay in the different systems. In addition to deriving the expected delay, we derive the expected cycle time and the expected number of customers found in a given station, for the exhaustive and the gated systems.

Three types of service policy are studied in this chapter: 1) Exhaustive service. 2) Gated service. 3) Non-exhaustive service. After a detailed description of the system model, done in section 4.2, the first three sections of this chapter deal with the exhaustive scheme. In section 4.3 we analyze the number of customers found in the exhaustive system at polling instants. In section 4.4 we analyze the length of the service periods and the the cycle length in the system. In section 4.5 we analyze the number of customers found in the system in arbitrary times, and the delay suffered by arbitrary customers. Since the analysis of the gated system is quite similar to that of the exhaustive system, we do this analysis in short, and devote section 4.6 for it. Section 4.7 contains the analysis of the expected delay in the non exhaustive scheme. In section 4.8 we compare the delay in the three different schemes, and also compare them to the delay in the equivalent cyclic polling schemes. Lastly, due to the many algebraic symbols used in our analysis, and to help in reading this chapter, we provide, in appendix D, a glossary of the notation used in this chapter. The rest of the appendices (A, B and C) are devoted to the algebraic derivation of some of the equations appearing in the text.

4.1.1 Previous Work

As already mentioned the amount of work done in the area of polling systems is tremendous. For this reason we will mention only the work that is strongly related to this chapter.

The discrete model of exhaustive-service cyclic-polling system with N stations and random switch over periods were studied by Konheim [Konh80], Swartz [Swar80], Rubin and DeMoraes [Rubi81, Mora81]. The discrete model of the gated-service cyclic polling system with N and random length switch over periods were studied in [Rubi81, Mora81]. The expected delay in a non exhaustive cyclic-polling system where at most one customer is served in a service period was studied, for a symmetric system in [Nomu78]. Many other references dealt with the cyclic polling systems but under different assumptions (continuous model, zero length switch over period, approximated results and so on).

Lastly, a recent paper aiming at tutorial of polling systems was written by Takagi and Kleinrock [Taka83]. This paper summarizes the known results for polling systems and brings a coherent and organized derivation of most of the known results. This paper served us as an excellent source for previous results, and helped us in the derivation of many of our results. Many of the references not mentioned here can be found in that paper.

4.2 Model Description

We consider a system with N infinite-buffer queues and one server. The time is slotted with slot size equals to the service time of a customer. The time interval $(t\text{-}1,t)$ is called the tth slot. Customers which arrive during the tth slot may be served at the $t+1$st slot.

The arrival process to each queue consists of bulks of customers. We denote by $X_i(t)$ the number of customers arriving during the tth slot, i.e., this is the size of the bulk arriving at station i during the tth slot. For each queue i, the arrival sequence, $\{X_i(t) : t=1,2,...\}$ is assumed to be an independent and identically distributed sequence of random variables. The z-transform, mean and variance of $X_i(t)$ are given by:

$$P_i(z) \triangleq E[z^{X_i(t)}] \qquad \mu_i \triangleq E[X_i(t)] = P_i^{(1)}(1)$$

$$\sigma_i^2 \triangleq Var[X_i(t)] = P_i^{(2)}(1)+P_i^{(1)}(1)-[P_i^{(1)}(1)]^2 \tag{4.1}$$

The polling policy is the following: after completing the service of queue i (the period during which the server serves a queue is called *a service period*), the server goes for a *switch-over period.* During this period, none of the queues is served, and it may be considered as the time required to switch from queue i to the next queue to be served. The switch-over period is associated with the queue previously served (in this case i), and its length (distribution) may be a function of i. At the end of the switch-over period, the server picks, in a random fashion, the next queue to be served. The *polling policy* is that at this moment queue j is selected to be served with probability p_j. It is also assumed that the p_i's add up to one, i.e.,

$$\sum_{i=1}^{N} p_i = 1$$

After serving queue j, the server, again, will go for a switch-over period (associated with queue j) and then pick, in a random fashion, the next queue to be served.

Three types of *service policies* are considered in this chapter: the *exhaustive policy,* the *gated policy,* and the *non-exhaustive policy.* In the *exhaustive* policy, when queue i is selected to be served, the server will serve this queue until the queue becomes empty. Thus, all customers found in the queue at the beginning of the service period, and those who arrived during the service period are served in that service period. In the *gated* policy when queue i is selected to be served, the server will serve, in that service period, all (and only) the customers found in queue i at the beginning of the service period. Thus, none of the customers arriving during the service period, will be served during this period. In the *non-exhaustive* policy when queue i is selected to be served, the server will serve at this service period exactly one customer from queue i. Naturally, if no customer is found in this queue, when it is polled, no service is given to the queue in this service period.

Three types of epochs are of interest: the time at which the server starts serving queue i in the mth time, the time at which this service period ends, and the time when the switch-over period, succeeding this service period, terminates. The mth period at which queue i is served is called *the mth service period of queue i.* The switch over period succeeding the mth service period of queue i is called *the mth switch over period of queue i.* Let us use the following notation:

$\underline{\tau}_i(m) \triangleq$ the instant at which the mth service period of queue i starts.

$\tau_i(m) \triangleq$ the instant at which the mth service period of queue i terminates.

$\overline{\tau}_i(m) \triangleq$ the instant at which the mth switch-over period of queue i terminates.

From this notation and the service policy described above, it is obvious, that for each i and m there exist j and n such that $\overline{\tau}_i(m)=\underline{\tau}_j(n)$.

The length of the mth switch-over period of queue i is $\bar{\tau}_i(m)-\tau_i(m)$. For each queue it is assumed that the sequence of switch-over periods associated with it, $\{\bar{\tau}_i(m)-\tau_i(m) : m=1,2,...\}$, is a sequence of independent and identically distributed random variables. The z-transform mean and variance of $\bar{\tau}_i(m)-\tau_i(m)$ are given by:

$$R_i(z) \triangleq E[z^{\bar{\tau}_i(m)-\tau_i(m)}] \qquad r_i \triangleq E[\bar{\tau}_i(m)-\tau_i(m)] = R_i^{(1)}(1)$$

$$\delta_i^2 \triangleq Var[\bar{\tau}_i(m)-\tau_i(m)] = R_i^{(2)}(1)+R_i^{(1)}(1)-[R_i^{(1)}]^2 \tag{4.2}$$

In addition to the notation used above, let us denote by $\underline{\tau}(m)$ the instant at which the server starts polling at the mth time, independently of the index of the station polled. Similarly, $\tau(m)$ denotes the instant at which the server finishes serving at the mth time, and $\bar{\tau}(m)$ denotes the instant at which the server finishes the mth switch over period. According to this notation, it is clear that $\bar{\tau}(m)=\underline{\tau}(m+1)$.

4.3 Exhaustive System: Number of Customers at Polling Instants

We start our study by analyzing the number of customers found in the system at the polling instants. The number of customers found at the system is denoted as following:

$$L_i(t) \triangleq \text{number of customers at queue } i \text{ at time } t$$

$$L(t) \triangleq [L_1(t),L_2(t), \cdots ,L_N(t)]$$

While the process L(t) by itself is not a renewal process, it can be observed that the sequence $\underline{\tau}(1)$, $\underline{\tau}(2)$, $\underline{\tau}(3)$, $\cdots$ is a natural sequence of renewal epochs for the process L(t). Thus, if we observe the process L at the polling instants, $\{\underline{\tau}(m)\}$, the process is Markovian.

The z-transform of the number of customers found in the system at the mth polling instant is:

$$F_m(z_1,z_2, \cdots ,z_N) \triangleq E\left[\prod_{j=1}^{N} z_j^{L_j(\underline{\tau}(m))}\right] \tag{4.3}$$

The limiting z-transform when m approaches infinity is given by:

$$F(z_1,z_2, \cdots ,z_N) \triangleq \lim_{m\to\infty} F_m(z_1,z_2, \cdots ,z_N) \tag{4.4}$$

Similarly, the limiting marginal z-transform for $L_i(\underline{\tau}(m))$ when m approaches infinity is denoted by:

$$F_i(z) \triangleq \lim_{m\to\infty} E[z^{L_i(\underline{t}(m))}] = F(1,\cdots,1,z,1,\cdots,1) \quad (4.5)$$

In addition, let L_i^* be a random variable representing the number of customers at an arbitrary polling instant when the system is in equilibrium.

To calculate $F(z_1,z_2,\cdots,z_N)$ we express $F_{m+1}(z_1,z_2,\cdots,z_N)$ in terms of $F_m(z_1,z_2,\cdots,z_N)$. To do so, let us condition our calculation on the specific queue served during the mth service period. Let this queue be the ith queue.

The time interval of interest is the interval $[\underline{t}(m),\overline{\tau}(m)]$ which consists of the mth service period, $[\underline{t}(m),\tau(m)]$, and the mth switch over period, $[\tau(m),\overline{\tau}(m)]$. Since station i is the station served in the mth service period, there exists some (unique) n such that:

$$\underline{t}_i(n) = \underline{t}(m)\,, \qquad \tau_i(n) = \tau(m)\,, \qquad \overline{\tau}_i(n) = \overline{\tau}(m)\,,$$

Thus we are actually interested in the nth service period of station i and in the nth switch over period of queue i.

First, consider the service period for station i: $[\underline{t}_i(n),\tau_i(n)]$. The period starts when station i has $L_i(\underline{t}_i(n))$ customers in its queue and terminates when this queue is empty. The behavior of queue i during this period is equivalent to the behavior of the gambler's capital in the gambler's ruin problem (a short description of this problem is given in appendix A), where the initial capital is $L_i(\underline{t}_i(n))$. The length of the period, $\tau_i(n)-\underline{t}_i(n)$, corresponds to the gambler's ruin time. Thus, from the gambler's ruin problem we have:

$$E[w^{\tau_i(n)-\underline{t}_i(n)}] = H[\Theta_i(w)] \quad (4.6)$$

where

$$H(z) = E[z^{L_i(\underline{t}_i(n))}] \quad (4.7)$$

and $\Theta_i(w)$ satisfies

$$\Theta_i(w) - wP_i[\Theta_i(w)] = 0 \quad (4.8)$$

From (4.6) and (4.7) we get:

$$E[w^{\tau_i(n)-\underline{t}_i(n)}] = E[\{\Theta_i(w)\}^{L_i(\underline{t}_i(n))}] \quad (4.9)$$

and from (4.8) the first and the second derivatives of $\Theta_i(w)$, evaluated at $w=1$ are:

$$\Theta_i(1) = 1\,, \quad \Theta_i^{(1)}(1) = \frac{1}{1-\mu_i}\,, \quad \Theta_i^{(2)}(1) = \frac{\mu_i}{(1-\mu_i)^2} + \frac{\sigma_i^2}{(1-\mu_i)^3} \quad (4.10)$$

Now, let us return to calculate the number of customers in the system at the end of the service period. This is expressed as following:

$$L_j(r_i(n)) = \begin{cases} 0 & j=i \\ L_j(\underline{r}_i(n)) + \text{arrivals to queue } j \text{ in } [\underline{r}_i(n), r_i(n)] & j \neq i \end{cases} \quad (4.11)$$

The z-transform of the number of arrivals to all the stations (excluding i) during the interval $[\underline{r}_i(n), r_i(n)]$ is given by:

$$E\Big[\Big\{\prod_{j \neq i} P_j(z_j)\Big\}^{r_i(n)-\underline{r}_i(n)}\Big] \quad (4.12)$$

which is identical (see (4.9)) to:

$$E\Big[\Big\{\Theta_i\Big(\prod_{\substack{j=1 \\ (j \neq i)}}^{N} P_j(z_j)\Big)\Big\}^{L_i(\underline{r}_i(n))}\Big] \quad (4.13)$$

The z-transform of the number of customers in the system at the end of the service period is given by:

$$E\Big[\prod_{j=1}^{N} z_j^{L_j(r_i(n))}\Big] = E\Big[\prod_{\substack{j=1 \\ (j \neq i)}}^{N} z_j^{L_j(r_i(n))}\Big] = E\Big[\prod_{\substack{j=1 \\ (j \neq i)}}^{N} \Big\{z_j^{L_j(\underline{r}_i(n))}\{P_j(z_j)\}^{r_i(n)-\underline{r}_i(n)}\Big\}\Big] \quad (4.14)$$

From (4.12), (4.13) and (4.14), and since $\underline{r}_i(n) = \underline{s}(m)$, it can be shown, as shown in the analysis of exhaustive polling systems (see for example, [Konh80] and [Taka83]) that the z-transform of the number of customers at the end of the polling period is given by:

$$E\Big[\prod_{j=1}^{N} z_j^{L_j(r_i(n))}\Big] = F_m(z_1, z_2, \cdots, z_{i-1}, \Theta_i\Big(\prod_{\substack{j=1 \\ (j \neq i)}}^{N} P_j(z_j)\Big), z_{i+1}, \cdots, z_N) \quad (4.15)$$

Next, we calculate the number of customers arriving to the system during the nth switch over period of queue i, namely, during the interval $[r_i(n), \overline{r}_i(n)]$. This is simply the compound transform of the length of the switch over period and the number of arrivals during a slot:

$$R_i\Big(\prod_{j=1}^{N} P_j(z_j)\Big) \quad (4.16)$$

From (4.15), (4.16) and the fact that the number of arrivals during the switch over interval is independent of the number of customers in the system at the beginning of this interval, we get the z-transform of the number of customers in the system at the end of the switch over period. As we assumed at the beginning of this analysis, this z-transform is conditioned on the station served during the mth service period:

$$F_{m+1}(z_1, z_2, \cdots, z_N \mid i \text{ is served}) = R_i\Big(\prod_{j=1}^{N} P_j(z_j)\Big) \cdot F_m(z_1, z_2, \cdots, z_{i-1}, \Theta_i\Big(\prod_{\substack{j=1 \\ (j \neq i)}}^{N} P_j(z_j)\Big), z_{i+1}, \cdots, z_N) \quad (4.17)$$

Now, let us uncondition (4.17). From the system description, It is clear that station i is served in service period m with probability p_i. Thus, the unconditioned z-transform of the number in system at the end of the mth switch over period is:

$$F_{m+1}(z_1,z_2,\cdots,z_N) = p_1\cdot R_1\Big(\prod_{j=1}^{N}P_j(z_j)\Big)\cdot F_m(\Theta_1\Big(\prod_{\substack{j=1\\(j\neq 1)}}^{N}P_j(z_j)\Big),z_2,z_3,\cdots,z_N) \tag{4.18}$$

$$+\ p_2\cdot R_2\Big(\prod_{j=1}^{N}P_j(z_j)\Big)\cdot F_m(z_1,\Theta_2\Big(\prod_{\substack{j=1\\(j\neq 2)}}^{N}P_j(z_j)\Big),z_3,\cdots,z_N)$$

$$+\ \cdots$$

$$+\ p_N\cdot R_N\Big(\prod_{j=1}^{N}P_j(z_j)\Big)\cdot F_m(z_1,z_2,z_3,\cdots,\Theta_N\Big(\prod_{\substack{j=1\\(j\neq N)}}^{N}P_j(z_j)\Big))$$

Equation (4.18) expresses the z-transform of the number of customers in system at the beginning of the $m+1$st service period by the z-transform of the number of customers in the system at the beginning of the mth service period.

Now, in the limit, when m approaches infinity, and if the system is in equilibrium, we obtain:

$$F(z_1,z_2,\cdots,z_N) = p_1\cdot R_1\Big(\prod_{j=1}^{N}P_j(z_j)\Big)\cdot F(\Theta_1\Big(\prod_{\substack{j=1\\(j\neq 1)}}^{N}P_j(z_j)\Big),z_2,z_3,\cdots,z_N) \tag{4.19a}$$

$$+\ p_2\cdot R_2\Big(\prod_{j=1}^{N}P_j(z_j)\Big)\cdot F(z_1,\Theta_2\Big(\prod_{\substack{j=1\\(j\neq 2)}}^{N}P_j(z_j)\Big),z_3,\cdots,z_N)$$

$$+\ \cdots$$

$$+\ p_N\cdot R_N\Big(\prod_{j=1}^{N}P_j(z_j)\Big)\cdot F(z_1,z_2,z_3,\cdots,\Theta_N\Big(\prod_{\substack{j=1\\(j\neq N)}}^{N}P_j(z_j)\Big))$$

similarly, the conditional limiting z-transform is:

$$F(z_1,\cdots,z_N\ |i \text{ is previously served}) = R_i\Big(\prod_{j=1}^{N}P_j(z_j)\Big)\cdot F(z_1,z_2,\cdots,z_{i-1},\Theta_i\Big(\prod_{\substack{j=1\\(j\neq i)}}^{N}P_j(z_j)\Big),z_{i+1},\cdots,z_N)$$

(4.19b)

4.3.1 Number of Customers at Polling Instants: Mean and Variance

Following the approach used in the literature, we next compute from (4.19) the mean and variance of the number of customers found in the system at polling instants. Let L_i^* denote the number of customers at station i at polling instants. In addition, let us denote the partial derivatives of $F(z_1,z_2,\cdots,z_N)$ as following:

$$f(i) \triangleq \frac{\partial F(z_1,z_2,\cdots,z_N)}{\partial z_i}\Big|_{\vec{z}=1} \; ; \qquad i=1,2,\cdots,N \tag{4.20}$$

$$f(i,j) \triangleq \frac{\partial^2 F(z_1,z_2,\cdots,z_N)}{\partial z_i \partial z_j}\Big|_{\vec{z}=1} \; ; \qquad i,j=1,2,\cdots,N \tag{4.21}$$

where $\vec{z}$ is the vector of the z_i's, namely, $\vec{z} = (z_1,z_2,\cdots,z_N)$, and 1 (when applicable) corresponds to the vector (1, 1, $\cdots$, 1). Similarly, we define $f(i\,|k)$ and $f(i,j\,|k)$ to be the corresponding measures, conditioning on the station (k) served during the previous cycle:

$$f(i\,|k) \triangleq \frac{\partial F(z_1,z_2,\cdots,z_N\,|k)}{\partial z_i}\Big|_{\vec{z}=1} \; ; \qquad i=1,2,\cdots,N$$

$$f(i,j\,|k) \triangleq \frac{\partial^2 F(z_1,z_2,\cdots,z_N\,|k)}{\partial z_i \partial z_j}\Big|_{\vec{z}=1} \; ; \qquad i,j=1,2,\cdots,N$$

Using this notation, then:

$$E[L_i^*] = f(i)\,, \qquad Var[L_i^*] = f(i,i)+E[L_i^*]-\{E[L_i^*]\}^2 \tag{4.22}$$

In appendix A we differentiate (4.19) with respect to the z_i's to calculate the terms $f(i), i=1,2,\cdots,N$. This yields a set of N linear equations, of the form:

$$f(j) = \frac{\mu_j}{p_j}\cdot\Big(\sum_{i=1}^{N} p_i r_i + \sum_{\substack{i=1\\ i\neq j}}^{N} \frac{p_i f(i)}{1-\mu_i}\Big) \tag{4.23}$$

The solution of this equation set is shown (in Appendix A) to be:

$$E[L_j^*] = f(j) = \frac{(1-\mu_j)\mu_j \sum_{i=1}^{N} p_i r_i}{p_j\Big(1-\sum_{i=1}^{N}\mu_i\Big)} \tag{4.24}$$

which is the expected length of queue j at polling instants.

Comparing this result to the expected queue length in the regular exhaustive polling system, we note that when $p_i = 1/N$ for all i, the above expression becomes:

$$E[L_j^*] = f(j) = \frac{(1-\mu_j)\mu_j \sum_{i=1}^{N} r_i}{\left(1-\sum_{i=1}^{N}\mu_i\right)}$$

which is identical to the expected queue length at polling instants in the cyclic-polling exhaustive system.

In the case of symmetric system, i.e., when all the stations are identical, we have:

$$\mu_i = \mu\ , \qquad p_i = \frac{1}{N}\ , \qquad r_i = r\ ; \qquad\qquad i = 1, 2, \cdots, N$$

so (4.24) becomes:

$$E[L_j^*] = f(j) = \frac{Nr\mu(1-\mu)}{1-N\mu} \tag{4.25}$$

This is the expected value of the queue length at polling instants for the case of symmetric stations. The reader may note that this value is identical to the expected queue length (at polling instants where the stations are symmetric) in a cyclically exhaustive polling system (see for example [Taka83]).

The computation of $f(i,i)$ is more complicated than that of $f(i)$. In appendix A we differentiate (4.19) twice with respect to the z_i's. This yields a set of N^2 linear equations that can be solved by numerical methods. When the switch over period and the arrival process are assumed to be identical for all stations, this set becomes:

$$f(j,k) = \sum_{\substack{i=1 \\ (i\neq j) \\ (i\neq k)}}^{N} \Big(a + b[f(j)+f(k)] + cf(i) + d[f(i,j)+f(i,k)] + f(j,k) + d^2 f(i,i)\Big)\cdot p_i$$

$$+ \Big(a + b[f(j) + df(k)]\Big)\cdot p_k + \Big(a + b[f(k) + df(j)]\Big)\cdot p_j \qquad j\neq k \tag{4.26a}$$

$$f(j,j) = \sum_{i=1}^{N} p_i f(j,j \mid i) =$$

$$= \sum_{\substack{i=1\\(i\neq j)}}^{N} \{ a + r(\sigma^2-\mu) + 2bf(j) + \left(\frac{\sigma^2-\mu}{1-\mu} + c\right) f(i) + f(j,j) + 2df(i,j) + d^2 f(i,i)\} \cdot p_i$$

$$+ \; p_j[a + r(\sigma^2-\mu)] \qquad (4.26b)$$

Where the terms a, b, c, and d are defined in the appendix.

In the case of fully symmetric stations (namely, in a system where in addition to symmetric arrival process and symmetric switch over periods also the p_i's are identical for all stations) (4.26a) and (4.26b) can be solved (see appendix A) analytically. This yields the following solution for $f(i,i)$, $i=1, \cdots N$:

$$f(i,i) = \frac{\delta^2\mu^2N(1-\mu)}{1-N\mu} + \frac{\sigma^2 rN[1-(N+1)\mu+(2N-1)\mu^2]}{(1-N\mu)^2} - \frac{Nr\mu(1-\mu)}{1-N\mu}$$

$$+ \frac{N^2r^2\mu^2(1-\mu)^2}{(1-N\mu)^2} + \frac{Nr^2\mu^2(N-1)(1-\mu)}{(1-N\mu)^2} \qquad (4.27)$$

From (4.22) and (4.27) we can now calculate the second moment and the variance of the number of customers at polling instants:

$$E[\{L_i^*\}^2] = \frac{\delta^2\mu^2N(1-\mu)}{1-N\mu} + \frac{\sigma^2 rN[1-(N+1)\mu+(2N-1)\mu^2]}{(1-N\mu)^2} \qquad (4.28)$$

$$+ \frac{N^2r^2\mu^2(1-\mu)^2}{(1-N\mu)^2} + \frac{Nr^2\mu^2(N-1)(1-\mu)}{(1-N\mu)^2}$$

$$Var[L_i^*] = \frac{\delta^2\mu^2N(1-\mu)}{1-N\mu} + \frac{\sigma^2 rN[1-(N+1)\mu+(2N-1)\mu^2]}{(1-N\mu)^2} + \frac{Nr^2\mu^2(N-1)(1-\mu)}{(1-N\mu)^2} \qquad (4.29)$$

4.4 Exhaustive System: Service-Time, Intervisit Time and Cycle Time

In this section we calculate the length of the different periods observed in the system. Let us define as the *service period of queue i*, S_i, the period at which queue i is served. *The intervisit period of queue i*, I_i, is defined to be the period in between two consecutive services of queue i. *A cycle of queue i*, C_i, consists of a service period followed by an intervisit period. The length of a service period, S_i is given by $\tau_i(m)-\underline{\tau}_i(m)$, the length of an intervisit period is given by $\bar{\tau}_i(m)-\tau_i(m)$, and the length of the cycle is given by $\bar{\tau}_i(m)-\underline{\tau}_i(m)$. These measures are called the *service time*, the *intervisit time*, and the *cycle time*, respectively. In addition we define:

$$S_i(z) \triangleq E[z^{\tau_i(m)-\underline{\tau}_i(m)}] \ , \qquad I_i(z) \triangleq E[z^{\bar{\tau}_i(m)-\tau_i(m)}] \ , \qquad C_i(z) \triangleq E[z^{\bar{\tau}_i(m)-\underline{\tau}_i(m)}]$$

It is easy to realize that the cyclic behavior of our system is very similar to that of the system where the queues are served in cyclic fashion (polling system). In both systems a service period of queue i is followed by an intervisit period of queue i, and this is followed by a service period of queue i. The only difference between the systems is the specific behavior during the periods. For this reason, it is rather natural for us to follow, in the sequel, the approach used in [Taka83] to analyze the cycle time of the polling system.

To analyze the service time of queue i we recall (4.9). This can be rewritten as:

$$E[z^{\tau_i(m)-\underline{\tau}_i(m)}] = E[\{\Theta_i(z)\}^{L_i(\underline{\tau}_i(m))}] \tag{4.30}$$

where $\Theta(z)$ is given in (4.8). Now, using (4.10) and (4.30) it is easy to derive the expected value and the variance of S_i in terms of the expected value and variance of L_i^*:

$$E[S_i] = E[\tau_i(m)-\underline{\tau}_i(m)] = E[L_i^*]\cdot\Theta_i^{(1)}(1) \tag{4.31a}$$

$$Var[S_i] = Var[\tau_i(m)-\underline{\tau}_i(m)] = \frac{Var[L_i^*]}{(1-\mu_i)^2}+\frac{\sigma_i^2 E[L_i^*]}{(1-\mu_i)^3} \tag{4.31b}$$

Now, using (4.24) we get:

$$E[S_i] = \frac{\mu_i\sum_{j=1}^{N}p_j r_j}{p_i\left(1-\sum_{j=1}^{N}\mu_j\right)} \tag{4.32a}$$

$$Var[S_i] = \frac{1}{(1-\mu_i)^2}\left(Var[L_i^*]+\frac{\sigma_i^2\mu_i\sum_{j=1}^{N}p_j r_j}{p_i\left(1-\sum_{j=1}^{N}\mu_j\right)}\right) \tag{4.32b}$$

In the case of symmetric stations we have:

$$E[S_i] = \frac{Nr\mu}{1-N\mu} \tag{4.33a}$$

$$Var[S_i] = \frac{1}{(1-\mu)^2}\cdot\left[\frac{\delta^2\mu^2N(1-\mu)}{1-N\mu}+\frac{Nr\sigma^2[1-N\mu+(N-1)\mu^2]}{(1-N\mu)^2}+\frac{Nr^2\mu^2(N-1)(1-\mu)}{(1-N\mu)^2}\right] \tag{4.33b}$$

Next we calculate the intervisit time of queue i. To do so we relate the number of customers found at queue i when it is polled (namely, at time $\tau_i(m)$) to the length of the previous intervisit period. This relation is:

$$E[z^{L_i(\bar{\tau}_i(m))}] = E[\{P_i(z)\}^{\bar{\tau}_i(m)-\tau_i(m)}] \tag{4.34}$$

From (4.34) it is easy to find (see [Taka83]) the expected value and the variance of I_i:

$$E[L_i^*] = \mu_i E[I_i] \ , \qquad Var[L_i^*] = \mu_i^2 Var[I_i] + \sigma_i^2 E[I_i] \tag{4.35}$$

and substituting (4.24) in (4.35) we get:

$$E[I_i] = \frac{(1-\mu_i)\sum_{j=1}^{N} p_j r_j}{p_i\left(1-\sum_{j=1}^{N}\mu_j\right)} \ , \qquad Var[I_i] = \frac{1}{\mu_i^2}\cdot\left(Var[L_i^*] - \frac{(1-\mu_i)\sigma_i^2\sum_{j=1}^{N} p_j r_j}{p_i\left(1-\sum_{j=1}^{N}\mu_j\right)}\right) \tag{4.36}$$

In the case of symmetric stations we get:

$$E[I_i] = \frac{Nr(1-\mu)}{1-N\mu} \ , \qquad Var[I_i] = \frac{N\delta^2(1-\mu)}{1-N\mu} + \frac{Nr\sigma^2(N-1)}{(1-N\mu)^2} + \frac{Nr^2(N-1)(1-\mu)}{(1-N\mu)^2} \tag{4.37}$$

Lastly we calculate the length of the cycle time. The expected value of the cycle time of queue i is simply the sum of the appropriate expected service time and the expected intervisit time (even though these variables are not independent):

$$E[C_i] = E[S_i] + E[I_i] = \frac{\sum_{j=1}^{N} p_j r_j}{p_i\left(1-\sum_{j=1}^{N}\mu_j\right)} \tag{4.38}$$

The computation of $Var[C_i]$ is more difficult. This can be done by relating the length of the cycle time to the length of the intervisit time and to $\Theta_i(z)$. In [Taka83] it is shown that these are related as following:

$$C_i(z) = I_i[\Theta_i(z)] \tag{4.39}$$

From (4.39) it is now easy to calculate $Var[C_i]$:

$$Var[C_i] = \frac{1}{\mu_i^2(1-\mu_i)^2}\cdot\left[Var[L_i^*] + \frac{\sigma_i^2(\mu_i^2+\mu_i-1)\sum_{j=1}^{N} p_j r_j}{p_i\left(1-\sum_{j=1}^{N}\mu_j\right)}\right] \tag{4.40}$$

In the case of symmetric stations we have:

$$E[C_i] = \frac{Nr}{1-N\mu} , \qquad Var[C_i] = \frac{1}{1-\mu} \left\{ \frac{N\delta^2}{1-N\mu} + \frac{(N-1)Nr^2}{(1-N\mu)^2} + \frac{N^2\sigma^2 r}{(1-N\mu)^2} \right] \tag{4.41}$$

4.5 Exhaustive System: Number of Customers at Arbitrary Times and Waiting Times

Let us define $Q_i(z)$ to be the z-transform of the number of customers found at queue i at arbitrary times:

$$Q_i(z) \triangleq E[z^{L_i}]$$

This z-transform can be computed as the time average of $z^{L_i(t)}$ over the average cycle time:

$$Q_i(z) = \frac{E\left[\sum_{t=\underline{\tau}_i(m)}^{\overline{\tau}_i(m)-1} z^{L_i(t)}\right]}{E[\overline{\tau}_i(m)-\underline{\tau}_i(m)]} \tag{4.42}$$

where the denominator is the expected length of the cycle time (given by (4.41)).

To calculate the numerator of (4.42) we divide the sum to two parts: one corresponds to the service period and the other corresponds to the intervisit period:

$$\sum_{t=\underline{\tau}_i(m)}^{\overline{\tau}_i(m)-1} z^{L_i(t)} = \sum_{t=\underline{\tau}_i(m)}^{\tau_i(m)-1} z^{L_i(t)} + \sum_{t=\tau_i(m)}^{\overline{\tau}_i(m)-1} z^{L_i(t)}$$

The expected value of the first sum is shown in [Taka83] to be:

$$\sum_{t=\underline{\tau}_i(m)}^{\tau_i(m)-1} z^{L_i(t)} = z\frac{F_i(z)-1}{z-P_i(z)} \tag{4.43}$$

The expected value of the second sum is shown in [Taka83] to be:

$$\sum_{t=\tau_i(m)}^{\overline{\tau}_i(m)-1} z^{L_i(t)} = \frac{1-F_i(z)}{1-P_i(z)} \tag{4.44}$$

Thus, from (4.42), (4.43) and (4.44) we get:

$$Q_i(z) = \frac{p_i\left(1-\sum_{j=1}^{N}\mu_j\right)}{\sum_{j=1}^{N} p_j r_j} \cdot \left[z\frac{F_i(z)-1}{z-P_i(z)} + \frac{1-F_i(z)}{1-P_i(z)} \right] \tag{4.45}$$

To evaluate the expected value of L_i we differentiate (4.45). This yields:

$$\frac{\partial}{\partial z}\left[\frac{F_i(z)-1}{P_i(z)-1}\right]_{z=1} = \frac{\mu_i E[\{L_i^*\}^2]-(\sigma_i^2+\mu_i^2)E[L_i^*]}{2\mu_i^2} \tag{4.46}$$

$$\frac{\partial}{\partial z}\left[z\frac{F_i(z)-1}{z-P_i(z)}\right]_{z=1} = \frac{\sigma_i^2 E[L_i^*]}{2(1-\mu_i)^2} + \frac{E[L_i^*]}{2} + \frac{E[\{L_i^*\}^2]}{2(1-\mu_i)} \tag{4.47}$$

From (4.46), (4.47) and the expected cycle length we get:

$$E[L_i] = \frac{E[\{L_i^*\}^2]}{2E[L_i^*]} + \frac{\sigma_i^2}{2}\left(\frac{1}{1-\mu_i} - \frac{1}{\mu_i}\right) \tag{4.48}$$

In the case of identical stations, we use (4.24) and (4.28) to get:

$$E[L_i] = \frac{1}{2}\cdot\left[\frac{\delta^2\mu}{r} + \frac{\sigma^2}{1-N\mu} + \frac{Nr\mu(1-\mu)}{1-N\mu} + \frac{(N-1)r\mu}{1-N\mu}\right] \tag{4.49}$$

Next we calculate the waiting times and system times observed in the system. Let C_j be an arbitrary customer. Recalling that customers arrive to the system in bulks, we realize that the waiting time of C_j consists of the sum of two independent random variables:

1. The waiting time of the first customer in the bulk in which C_j arrives.

2. The service time of all the customers which arrive together with C_j (the same bulk) and which are served ahead of C_j.

Let W_i denote the waiting time (excluding service time) of the first customer served in a bulk (for a bulk that arrives to queue i) and let $W_i(z)$ be the z-transform of W_i. Let V_i be the number of customers which arrive together with C_j to queue i (in the same bulk) and which are served before C_j, and let $V_i(z)$ be the z-transform of V_i. Let T_i denote the system time (waiting time plus service time) of an arbitrary customer, and $T_i(z)$ denote the z-transform of T_i.

From the description given above, and since the service time of every customer is one unit of time, it is clear that the z-transform of the system time observed by arbitrary customers at queue i is:

$$T_i(z) = z \cdot W_i(z) \cdot V_i(z)$$

It is straight forward to calculate $V_i(z)$ from the z-transform of the bulk size, $P_i(z)$, and from its first moment, μ_i. This is shown in [Taka83] to be:

$$V_i(z) = \frac{1-P_i(z)}{\mu_i(1-z)} \tag{4.50}$$

Also in [Taka83] it is shown that in an exhaustive system $W_i(z)$ can be calculated from the z-transform of the idle period length, $I_i(z)$, and from the expected cycle length, $E[C_i]$, as following:

$$W_i(z) = \frac{z}{E[C_i]} \cdot \frac{I_i(z)-1}{z-P_i(z)}$$

In a careful examination of the derivation of this expression we realize that it was derived under the assumption that a customer which arrived at slot t can be served at slot t. Since in our model a customer arriving at time t can only be served at time $t+1$, this expression should be corrected (for our model) to:

$$W_i(z) = \frac{1}{P_i(z)} \cdot \frac{z}{E[C_i]} \cdot \frac{I_i(z)-1}{z-P_i(z)} \tag{4.51}$$

Since this result does not depend on the specific order of service selected by the server (in [Taka83] it is cyclic order, and here it is random order) and only depends on the fact that the service is exhaustive, we can apply it to our system. Thus using (4.38) and (4.10) we get $W_i(z)$ in terms of $I_i(z)$ and the system parameters:

$$W_i(z) = \frac{1}{P_i(z)} \cdot \frac{\sum_{j=1}^{N} p_j r_j}{p_i\left(1-\sum_{j=1}^{N}\mu_j\right)} \cdot z\,\frac{I_i(z)-1}{z-P_i(z)} \tag{4.52}$$

Since $I_i(z)$ can be calculated from $F_i(z)$, this equation actually expresses $W_i(z)$ in terms of $F_i(z)$.

Now, to calculate the expected system time of an arbitrary customer which arrives to queue i, we can either differentiate $z \cdot W_i(z) \cdot V_i(z)$ with respect to z and evaluate at $z=1$, or to apply Little's result [Litt61] to (4.48) and (4.49). Selecting the second approach, we finally get the expected waiting time for an arbitrary customer arriving to queue i:

$$E[T_i] = E[W_i]+E[V_i]+1 = \frac{E[L_i]}{\mu_i} = \frac{E[(L_i^*)^2]}{2\mu_i E[L_i^*]}+\frac{\sigma_i^2}{2\mu_i}\left(\frac{1}{1-\mu_i}-\frac{1}{\mu_i}\right) \tag{4.53}$$

which in the case of symmetric stations becomes:

$$E[T_i] = \frac{1}{2}\cdot\left[\frac{\delta^2}{r}+\frac{\sigma^2}{(1-N\mu)\mu}+\frac{Nr(1-\mu)}{1-N\mu}+\frac{(N-1)r}{1-N\mu}\right] \tag{4.54}$$

4.6 Random-Polling Gated-Service Policy

The service policy considered in this section is the *random-polling gated-service* policy. According to this policy, when the server polls station i, it serves only the customers found at queue i at the polling instant. Customers who arrive to station i during the service period of this station will not be served during that service period. Rather, they will wait and get served during the next service period of station i.

The polling policy, for this system, is the same as considered in the previous model, namely, a random polling policy. Thus, at a given polling instant, station i is polled with probability p_i.

The system model, as in the previous sections, is a discrete time model, and the arrival process is the same as in the previous model. The service time of a customer equals, as before, the length of a time slot. Due to the similarity of this model to the exhaustive model we keep our notation the same.

Like in the analysis of the exhaustive system, the key of this analysis is the z-transform of the number of customers found in the system at the end of a switch-over period. This is $F_m(z_1,z_2,\cdots,z_N)$, as defined in (4.3).

For the gated policy the length of the service period of station i, is simply the number of customers found in queue i at the polling instant:

$$\tau_i(m)-t_i(m) = L_i(t_i(m)) \tag{4.55}$$

Thus, the z-transform of the number of customers arriving during this period is given by:

$$E\left[\prod_{j=1}^{N}\{P_j(z_j)\}^{\tau_i(m)-t_i(m)}\right] = E\left[\prod_{j=1}^{N}\{P_j(z_j)\}^{L_i(t_i(m))}\right] \tag{4.56}$$

Thus, (4.17) is replaced by:

$$F_{m+1}(z_1,z_2,\cdots,z_N \mid i \text{ is served}) = R_i\Big(\prod_{j=1}^{N} P_j(z_j)\Big)\cdot F_m(z_1,z_2,\cdots,z_{i-1},\prod_{\substack{j=1\\(j\neq i)}}^{N} P_j(z_j),z_{i+1},\cdots,z_N) \qquad (4.57)$$

and (4.19a) and (4.19b) are replaced by:

$$F(z_1,z_2,\cdots,z_N) = p_1\cdot R_1\Big(\prod_{j=1}^{N} P_j(z_j)\Big)\cdot F(\prod_{\substack{j=1\\(j\neq 1)}}^{N} P_j(z_j),z_2,z_3,\cdots,z_N)$$

$$+\, p_2\cdot R_2\Big(\prod_{j=1}^{N} P_j(z_j)\Big)\cdot F(z_1,\prod_{\substack{j=1\\(j\neq 2)}}^{N} P_j(z_j),z_3,\cdots,z_N)$$

$$+\cdots$$

$$+\, p_N\cdot R_N\Big(\prod_{j=1}^{N} P_j(z_j)\Big)\cdot F(z_1,z_2,z_3,\cdots,\prod_{\substack{j=1\\(j\neq N)}}^{N} P_j(z_j)) \qquad (4.58a)$$

$$F(z_1,\cdots,z_N \mid i \text{ is previously served}) = R_i\Big(\prod_{j=1}^{N} P_j(z_j)\Big)\cdot F(z_1,z_2,\cdots,z_{i-1},\prod_{\substack{j=1\\(j\neq i)}}^{N} P_j(z_j),z_{i+1},\cdots,z_N) \qquad (4.58b)$$

Defining the moments of L_i^*, ($f(i)$, $f(i,j)$, $f(i \mid k)$, $f(i,j \mid k)$) as in the exhaustive model and differentiating (4.58b), we get the following set of equations:

$$f(i \mid i) = r_i\mu_i + \mu_i f(i) \qquad (4.59a)$$

$$f(j \mid i) = r_i\mu_j + \mu_j f(i) + f(j) \qquad i\neq j \qquad (4.59b)$$

Summing (4.59a) and (4.59b) gives:

$$f(j) = \sum_{i=1}^{N} p_i f(j \mid i) = \sum_{i=1}^{N} p_i\Big(r_i\mu_j + \mu_j f(i)\Big) + \sum_{\substack{i=1\\(i\neq j)}}^{N} p_i f(j) \qquad (4.60)$$

This equation is solved in appendix B to yield:

$$E[L_j^*] = f(j) = \frac{\mu_j}{p_j}\cdot\frac{\sum_{i=1}^{N} p_i r_i}{1-\sum_{i=1}^{N}\mu_i} \qquad (4.61)$$

When $p_i = 1/N$ for every i this becomes:

$$E[L_j^*] = f(j) = \frac{\mu_j \cdot \sum_{i=1}^{N} r_i}{1 - \sum_{i=1}^{N} \mu_i}$$

which is identical to the equivalent expression in the gated system where the polling is done in a cyclic fashion. In the case of fully symmetric stations $E[L_j^*]$ is:

$$E[L_j^*] = f(j) = \frac{Nr\mu}{1-N\mu} \tag{4.62}$$

To find the variance of L_i^* we differentiate (4.58b) twice. This gives the following equation set:

$$f(j,k \mid i) = \mu_j\mu_k(\delta_i^2 + r_i^2) + r_i\mu_k f(j) + r_i\mu_j f(k) + f(i)\mu_j\mu_k(2r_i+1) + f(j,k)$$

$$+ \mu_j f(i,k) + \mu_k f(i,j) + \mu_j\mu_k f(i,i) \qquad i \neq j,\ i \neq k,\ j \neq k \tag{4.63a}$$

$$f(j,j \mid i) = \mu_j^2(\delta_i^2 + r_i^2) + r_i(\sigma_j^2 - \mu_j) + 2r_i\mu_j f(j) + f(i)[\sigma_j^2 - \mu_j + \mu_j^2(2r_i+1)]$$

$$+ f(j,j) + 2\mu_j f(i,j) + \mu_j^2 f(i,i) \qquad i \neq j \tag{4.63b}$$

$$f(j,k \mid j) = \mu_j\mu_k(\delta_j^2 + r_j^2) + r_j\mu_j f(k) + f(j)\mu_j\mu_k(2r_j+1) + \mu_j f(j,k) + \mu_j\mu_k f(j,j)$$

$$j \neq k \tag{4.63c}$$

$$f(j,j \mid j) = \mu_j^2(\delta_j^2 + r_j^2) + r_j(\sigma_j^2 - \mu_j) + f(j)[\sigma_j^2 - \mu_j + \mu_j^2(2r_j+1)] + \mu_j^2 f(j,j) \tag{4.63d}$$

Now, considering the following relations,

$$f(j,k) = \sum_{i=1}^{N} p_i \cdot f(j,k \mid i)$$

then we have a set of N^2 linear equations that can be solved by numerical methods to yield the solution of $f(i,i)$ for $i=1,2,\cdots,N$.

In appendix B we solve this set of equations for the case of symmetric stations. The solution is:

$$f^{(2)} = \frac{\sigma^2 rN[1-(N-1)\mu]}{(1+\mu)(1-N\mu)^2} + \frac{(\delta^2 - r^2)N\mu^2}{(1+\mu)(1-N\mu)} + \frac{(\mu+2r)N^2 r\mu^2}{(1+\mu)(1-N\mu)^2}$$

$$- \frac{\mu N r}{(1+\mu)(1-N\mu)} - \frac{\mu^2 N r}{(1+\mu)(1-N\mu)^2} \tag{4.64}$$

Now, using (4.62), (4.64) and the relation

$$Var[L_i^*] = f^{(2)} + f^{(1)} - \{f^{(1)}\}^2$$

we get:

$$Var[L_i^*] = \frac{\delta^2 \mu^2 N}{(1+\mu)(1-N\mu)} + \frac{\sigma^2 r N[1-(N-1)\mu]}{(1+\mu)(1-N\mu)^2} + \frac{(N-1)N\mu^2 r^2}{(1+\mu)(1-N\mu)^2} \tag{4.65}$$

This is the variance of the number of customers found in queue i at polling instants.

Now, that we have calculated the first two moments of the number of customers found in the system at polling instants we can calculate the cycle time, the number of customers found in the system at arbitrary moments and the waiting times.

Before we start the analysis, it should be noted that the cycle analysis of this system is independent of the polling order. This means that the way by which the variable representing the cycle time is related to the variable represting the number of customers found in the the system at polling instants is independent of the polling order. Thus, the cycle analysis of the cyclic-polling gated-service system and the cycle analysis of the random-polling gated-service system should be the same. For this reason, in the following analysis we can use most of the results derived in the literature for the discrete time cyclic-polling gated-service system.

To calculate the cycle time we note that in a gated system the z-transform of the cycle length is related to the z-transform of the number of customers found in queue i at polling instants as following:

$$F_i(z) = C_i[P_i(z)] \tag{4.66}$$

Thus, we can immediately calculate the mean and the variance of the cycle time:

$$E[C_i] = \frac{E[L_i^*]}{\mu_i} = \frac{1}{p_i} \cdot \frac{\sum_{j=1}^{N} p_j r_j}{1 - \sum_{i=1}^{N} \mu_i} \tag{4.67}$$

and:

$$Var[C_i] = \frac{Var[L_i^*]}{\mu_i^2} - \frac{E[L_i^*] \cdot \sigma_i^2}{\mu_i^3}$$

which, in the symmetric case becomes:

$$= \frac{N\delta^2}{(1+\mu)(1-N\mu)} + \frac{N^2\sigma^2 r}{(1+\mu)(1-N\mu)^2} + \frac{(N-1)Nr^2}{(1+\mu)(1-N\mu)^2} \tag{4.68}$$

Next we calculate the z-transform of the number of customers found in queue i at arbitrary moments. To do so, we borrow the following two relations from the analysis of the cyclic-polling gated-service systems (see [Taka83] for the derivation of these relations):

$$Q_i(z) = \frac{1}{E[C_i]} \cdot \frac{F_i[P_i(z)] - F_i(z)}{P_i(z) - z} \cdot \frac{(1-z)P_i(z)}{1-P_i(z)} \tag{4.69}$$

and:

$$E[L_i] = \frac{(1+\mu_i)E[\{L_i^*\}^2]}{2E[L_i^*]} - \frac{\sigma_i^2}{2\mu_i} \tag{4.70}$$

From (4.70) we can now calculate the expected value of the number of customers found in queue i at arbitrary moments. For a symmetric system this value is:

$$E[L_i] = \frac{\delta^2\mu}{2r} + \frac{\sigma^2}{2(1-N\mu)} + \frac{Nr\mu(1+\mu)}{2(1-N\mu)} + \frac{(N-1)r\mu}{2(1-N\mu)} \tag{4.71}$$

To calculate the waiting time in the system, we recall from the cyclic-polling gated-service system that the z-transform of the waiting time ($W_i(z)$) of the first customer in a bulk can be calculated from the z-transform of the cycle length ($C_i(z)$) and from the z-transform of the number of customers ($F_i(z)$) found in queue i at polling instants. The expression calculated in [Taka83] is:

$$W_i(z) = \frac{z[C_i(z) - F_i(z)]}{E[C_i] \cdot (z - P_i(z))}$$

which after the correction for our model (see comment with respect to equation (4.10)) becomes:

$$W_i(z) = \frac{1}{P_i(z)} \cdot \frac{z[C_i(z) - F_i(z)]}{E[C_i] \cdot (z - P_i(z))} \tag{4.72}$$

from which the expected waiting time of a first customer in a bulk can easily be calculated.

Now, let us calculate the expected system time of an arbitrary customer. We recall from the analysis of the exhaustive system that the waiting time of an arbitrary customer consists of the sum of two variables: W_i, the waiting time of the first customer in a bulk and V_i, the number of customer who arrive with the (arbitrary) tagged customer in the same bulk and served ahead of him. The expected value of the system time ($E[T_i]$) can be found by calculating

$E[W_i]$ from (4.67) and calculating $E[V_i]$ from (4.50) (recall that $V_i(z)$ is just a property of the arrival process and not of the service policy). We, instead, choose to use Little's result and apply it to (4.71). Thus, the system time of an arbitrary customer in a symmetric system is:

$$E[T_i] = \frac{\delta^2}{2r} + \frac{\sigma^2}{2\mu(1-N\mu)} + \frac{Nr(1+\mu)}{2(1-N\mu)} + \frac{(N-1)r}{2(1-N\mu)} \tag{4.73}$$

4.7 Random-Polling Non-Exhaustive Service Policy

The service policy considered in this section is the *random-polling non-exhaustive* policy. According to this policy, when the server polls station i, it serves only one customer from queue i.

The polling policy, for this system, is the same as considered in the previous models, namely, a random polling policy. Thus, at a given polling instant, station i is polled with probability p_i.

The system model, as before, is a discrete time model, and the arrival process is the same as in the previous models. The service time of a customer equals, as before, the length of a time slot. Most of the notations, from the previous sections, are kept the same, unless explicitly stated.

Like in the previous analysis the key of this analysis is the z-transform of the number of customers found in the system at *polling instants*. This is $F_m(z_1,z_2, \cdots ,z_N)$, as defined in (4.3).

To express $F_{m+1}(z_1,z_2, \cdots ,z_N)$ in terms of $F_m(z_1,z_2, \cdots ,z_N)$, we condition $F_{m+1}(z_1,z_2, \cdots ,z_N)$ on the station polled during the mth cycle. This expression is the following:

$$F(z_1,z_2, \cdots ,z_N \mid i \text{ is served }) = R_i\left(\prod_{j=1}^{N} P_j(z_j)\right)\cdot\left(\prod_{j=1}^{N} P_j(z_j)\right)\cdot\frac{1}{z_i}\left[F(z_1,z_2, \cdots ,z_N)-F(z_1,\cdots,0,\cdots z_N)\right]$$

$$+ {}^{*}R_i\left(\prod_{j=1}^{N} P_j(z_j)\right)\cdot F(z_1,\cdots,0,\cdots z_N) \tag{4.74}$$

Where $F(z_1,\cdots,0,\cdots z_N)$ is $F(z_1,z_2, \cdots ,z_N)$ where the ith element equals zero. The first term of this expression represents the situation where queue i is not empty when polled, so queue j "builds up" during the service slot, by a factor of $P_j(z_j)$, and one customer is removed from the ith buffer. The second term represents the situation where queue i is empty when polled, so no service period follows this polling instant. In both terms the factor $R_i\left(\prod_{j=1}^{N} P_j(z_j)\right)$ represents the queueing build up during the switch-over period prior to the $m+1$st polling instant.

From (4.74), and under equilibrium conditions, we get the following relation:

$$F(z_1,z_2,\cdots,z_N) =$$

$$p_1 \cdot R_1\left(\prod_{j=1}^{N} P_j(z_j)\right)\cdot\left[\left(\prod_{j=1}^{N} P_j(z_j)\right)\cdot F(z_1,z_2,\cdots,z_N)\cdot\frac{1}{z_1}+\left(1-\frac{1}{z_1}\prod_{j=1}^{N} P_j(z_j)\right)F(0,z_2,\cdots,z_N)\right]$$

$$+\ p_2 \cdot R_2\left(\prod_{j=1}^{N} P_j(z_j)\right)\cdot\left[\left(\prod_{j=1}^{N} P_j(z_j)\right)\cdot F(z_1,z_2,\cdots,z_N)\cdot\frac{1}{z_2}+\left(1-\frac{1}{z_2}\prod_{j=1}^{N} P_j(z_j)\right)F(z_1,0,z_3,\cdots,z_N)\right]$$

$$+ \cdots$$

$$+\ p_N \cdot R_N\left(\prod_{j=1}^{N} P_j(z_j)\right)\cdot\left[\left(\prod_{j=1}^{N} P_j(z_j)\right)\cdot F(z_1,z_2,\cdots,z_N)\cdot\frac{1}{z_N}+\left(1-\frac{1}{z_N}\prod_{j=1}^{N} P_j(z_j)\right)F(z_1,z_2,\cdots,z_{N-1},0)\right] \quad (4.75)$$

In the following we are interested in analyzing a fully symmetric system. Assuming symmetry, and substituting $z_1=z_2=\cdots=z_N=z$ in (4.75) we get:

$$F(z,z,\cdots,z) = \frac{1}{z}R\left(\{P(z)\}^N\right)\cdot\{P(z)\}^N F(z,z,\cdots,z)+R\left(\{P(z)\}^N\right)\cdot\left(1-\frac{\{P(z)\}^N}{z}\right)\cdot F(0,z,z,\cdots,z) \quad (4.76)$$

where we have used the observation that $F(0,z,z,\cdots,z)=F(z,0,z,\cdots,z)=\cdots=F(z,z,\cdots,z,0)$ due to symmetry. From (4.76) we have:

$$F(z,z,\cdots,z) = \frac{R\left(\{P(z)\}^N\right)\cdot\left(z-\{P(z)\}^N\right)\cdot F(0,z,z,\cdots,z)}{z-R\left(\{P(z)\}^N\right)\cdot\{P(z)\}^N} \quad (4.77)$$

Next, we substitute into (4.75) $z_1=z$ and $z_2=z_3=\cdots=z_N=1$. This yields:

$$F(z,1,1,\cdots,1) = \frac{1}{N}\cdot\left[R\left(P(z)\right)\cdot P(z)\cdot F(z,1,1,\cdots,1)\frac{1}{z} + R\left(P(z)\right)\cdot\left(1-\frac{P(z)}{z}\right)\cdot F(0,1,1,\cdots,1)\right]$$

$$+\ \frac{N-1}{N}\left[R\left(P(z)\right)\cdot P(z)\cdot F(z,1,1,\cdots,1) + R\left(P(z)\right)\cdot\left(1-P(z)\right)\cdot F(z,0,1,1,\cdots,1)\right] \quad (4.78)$$

where we have used the symmetry observations:

$$F(0,1,1,\cdots,1) = F(1,0,1,\cdots,1) = \quad \cdots \quad = F(1,1,1,\cdots,1,0)$$

$$F(z,0,1,1,\cdots,1) = F(z,1,0,1,\cdot 1,1) = \quad \cdots \quad = F(z,1,1,\cdots,1,1,0)$$

$$p_1 = p_2 = \quad \cdots \quad = p_N = \frac{1}{N}$$

From (4.78) we get:

$$F(z,1,1,\cdots,1) = \frac{(N-1)zR\left(P(z)\right)\cdot\left(1-P(z)\right)F(z,0,1,1,\cdots,1)}{Nz-R\left(P(z)\right)\cdot P(z)\cdot\left(1+(N-1)z\right)} + \frac{R\left(P(z)\right)\cdot\left(z-P(z)\right)\cdot F(0,1,1,\cdots,1)}{Nz-R\left(P(z)\right)\cdot P(z)\cdot\left(1+(N-1)z\right)} \quad (4.79)$$

Our next step is to calculate the probability that an arbitrary queue is empty at polling instants. This probability is given by:

$$f_0 \triangleq F(0,1,1,\cdots,1)$$

In appendix C we use (4.77) to calculate f_0. This is found to be:

$$f_0 = \frac{1-N\mu-Nr\mu}{1-N\mu} \quad (4.80)$$

Next we calculate the expected queue length at polling instants. To do so, we use two simple relations. The first relation is:

$$\left.\frac{\partial F(z,z,\cdots,z)}{\partial z}\right|_{z=1} = N\cdot\left.\frac{\partial F(z,1,1,\cdots,1)}{\partial z}\right|_{z=1} \quad (4.81)$$

This relation simply states that (at polling instants) the expected number of customers in the whole system is N times the expected number of customers in queue i ($i=1,2,\cdots,N$). This observation is true due to symmetry. The second relation is:

$$\left.\frac{\partial F(0,z,z,\cdots,z)}{\partial z}\right|_{z=1} = (N-1)\cdot\left.\frac{\partial F(z,0,1,1,\cdots,1)}{\partial z}\right|_{z=1} \quad (4.82)$$

which is also true due to symmetry.

For convenience let us introduce additional notation:

$$f_1 \triangleq \left.\frac{\partial F(z,0,1,1,\cdots,1)}{\partial z}\right|_{z=1}$$

Now we differentiate (4.77) and (4.79) with respect to z and evaluate the derivative at $z=1$. Differentiating (4.77) and using (4.82) we show in appendix C that:

$$\left.\frac{\partial F(z,z,\cdots,z)}{\partial z}\right|_{z=1} = \frac{(N-1)(1-N\mu)f_1}{1-N\mu-Nr\mu} + \frac{Nr\sigma^2}{2(1-N\mu)(1-N\mu-Nr\mu)} + \frac{N^2\mu^2\delta^2}{2(1-N\mu-Nr\mu)} + \frac{Nr\mu}{2} \quad (4.83)$$

Differentiating (4.79) with respect to z, evaluating the derivative at $z=1$ and using (4.82) we show in appendix C that:

$$N \cdot \frac{\partial F(z,1,1,\cdots,1)}{\partial z}\bigg|_{z=1} = \frac{-(N-1)N\mu f_1}{1-N\mu-Nr\mu}$$

$$+ \frac{N\cdot\left(N^2r\mu^3+N\mu^2(1-N\mu)(\delta^2-r^2)-2Nr\mu^2+(\sigma^2+\mu)Nr\right)}{2(1-N\mu)(1-N\mu-Nr\mu)} \tag{4.84}$$

Now, using (4.81) we equate (4.83) to (4.84) and solve (see appendix C) for f_1:

$$f_1 = \frac{(\sigma^2+\mu)Nr}{2(1-N\mu)} \tag{4.85}$$

Substituting (4.85) back into (4.83) we finally get an expression for the expected queue length at polling instants:

$$\frac{\partial F(z,1,1,\cdots,1)}{\partial z}\bigg|_{z=1} = \frac{(N-1)(\sigma^2+\mu)r}{2(1-N\mu-Nr\mu)} + \frac{r\sigma^2}{2(1-N\mu)(1-N\mu-Nr\mu)} + \frac{N\mu^2\delta^2}{2(1-N\mu-Nr\mu)} + \frac{r\mu}{2} \tag{4.86}$$

Having calculated the expected queue length of queue i at polling instants, we next calculate the expected queue length of queue i right after a customers left this queue. Let us denote:

$$G_i \triangleq E[L_i(\text{ service completion time at queue } i\ )]$$

First, since the queue chosen to be polled at a given polling instant is independent of the system status, it is clear that:

$$E[L_i(\text{ service starting time at queue } i\ )] = \frac{1}{1-f_0}\cdot\frac{\partial F(z,1,1,\cdots,1)}{\partial z}\bigg|_{z=1}$$

Second, we have:

$$G_i = E[L_i(\text{ service starting time at queue } i\ )] + \mu - 1$$

Thus, from these two relations and from (4.80) and (4.86) we get:

$$G_i = \frac{(N-1)(\sigma^2+\mu)(1-N\mu)}{2N\mu(1-N\mu-Nr\mu)} + \frac{\sigma^2}{2N\mu(1-N\mu-Nr\mu)}$$

$$+ \frac{N\mu^2\delta^2}{2(1-N\mu-Nr\mu)} + \frac{\mu\delta^2}{2r} + \frac{1-N\mu}{2N} + \mu - 1 \tag{4.87}$$

Which is the expected queue length at station i right after a customer left this station.

Next, using G_i, we calculate the system time (waiting time plus service time) of an arbitrary customer. Let us recall that T_i denotes the system time of an arbitrary customer when the system is in equilibrium. To calculate $E[T_i]$ we investigate the number of customers left in queue i behind a tagged customer, let say C_j. These customers are of two types:

i. Customers which arrived together with C_j (in the same bulk) but were queued behind C_j.

ii. Customers arrived to queue i during the system time of C_j.

Let V_i be the number of customers arriving to queue i together with C_j (the same bulk) but queued behind C_j, then the following relation is a direct result of the above observation:

$$G_i = E[V_i] + \mu \cdot E[T_i] \tag{4.88}$$

In appendix C we show that for symmetric stations

$$E[V_i] = \frac{\sigma^2 + \mu^2 - \mu}{2\mu} \tag{4.89}$$

Thus, substituting (4.89) and (4.87) into (4.88) we finally get the expected waiting time in the system:

$$E[T_i] = \frac{\delta^2}{2r} + \frac{\sigma^2}{2\mu(1-N\mu-Nr\mu)} + \frac{Nr\sigma^2}{2\mu(1-N\mu-Nr\mu)}$$

$$+ \frac{(N-1)r}{2(1-N\mu-Nr\mu)} + \frac{N\delta^2\mu}{2(1-N\mu-Nr\mu)} \tag{4.90}$$

4.7.1 The Probability of an Empty Buffer

An important property of a queueing system is the times at which the system is empty. In this sub-section we are interested in calculating the probability that a buffer is empty at some specific instants.

In the analysis done above we have calculated the probability that a buffer is empty at polling instants. This probability is given by:

$$Pr[\text{ queue } i \text{ is empty at polling instants }] = F(0,1,1,\cdots,1) = \frac{1-N\mu-Nr\mu}{1-N\mu}$$

From this measure it is now easy to calculate the probability that a buffer is empty at switch-over times. A *switch-over time* is the instant at which a switch-over period starts. Thus, the mth switch-over time is denoted by $\tau(m)$.

Let us recall that $X_i(t)$ denotes the number of customers arriving to queue i at time t, and that $P_i(z)$ is the z-transform of $X_i(t)$. In addition, let us denote by s_0 the probability that buffer i is empty at switch-over times:

$$s_0 \triangleq Pr[L_i(\tau(m)) = 0] \quad ; \qquad i=1,2,\cdots,N$$

Since every polling instant is the end of a switch-over period, the probability that buffer i is empty at a polling instant is related to the probability that this buffer is empty at the beginning of the preceding switch-over period as following:

$$Pr[L_i(\tau(m)) = 0 \mid \sigma(m),\, \tau(m)] \;=\; Pr[L_i(\sigma(m)) = 0]\cdot\left(\prod_{t=\sigma(m)+1}^{\tau(m)} Pr[X_i(t) = 0]\right) \qquad (4.91)$$

Now, since the arrival process to station i is independent of t, and since in the symmetric case it is also independent of i we can use the following notation:

$$x_0 \triangleq Pr[X_i(t) = 0] \quad ; \qquad i=1,2,\cdots,N$$

Thus (4.91) becomes:

$$Pr[L_i(\tau(m)) = 0 \mid \sigma(m),\, \tau(m)] \;=\; Pr[L_i(\sigma(m)) = 0] \cdot x_0^{[\tau(m)-\sigma(m)]}$$

or:

$$Pr[L_i(\tau(m)) = 0] \;=\; Pr[L_i(\sigma(m)) = 0]\cdot R(x_0) \qquad (4.92)$$

and finally, from (4.92) and (4.80) we have the probability that a buffer is empty at an arbitrary switch-over instant:

$$Pr[\text{ queue } i \text{ is empty at switch-over instants }] \;=\; \frac{1-N\mu-Nr\mu}{(1-N\mu)\cdot R(x_0)} \qquad (4.93)$$

4.8 Comparison of the Results and Discussion

In this section we compare the results derived in the previous sections for the expected system time in the different systems. These results are also compared to the expected delay observed in the equivalent cyclic-polling systems.

Before making the comparison, we should recall that they are differences, regarding the time at which a new arriving customer can be served, between the model used here to the model used in [Taka83]. For this reason the results derived in [Taka83] for the waiting time can be interpreted, in our model, as the system time. Thus, in the following, we will recall the results derived in [Taka83] for the waiting time in the different systems, and use them as they were the system time under our model.

Considering the cyclic polling system (in a discrete model, identical to our model), we recall that the expected waiting time (see [Taka83]) is:

$$\frac{\delta^2}{2r}+\frac{\sigma^2}{2\mu(1-N\mu)}+\frac{Nr(1-\mu)}{2(1-N\mu)}$$

The expected waiting time in the cyclic polling system under gated service policy is:

$$\frac{\delta^2}{2r}+\frac{\sigma^2}{2\mu(1-N\mu)}+\frac{Nr(1+\mu)}{2(1-N\mu)}$$

Considering the non-exhaustive cycling-polling policy with at most one customer served in a service period, we do not know about any previous study that considered a discrete time model. Thus the result for this type of system is not available for comparison.

These results and the results derived in this chapter are presented in table 1.

Comparing the system time in the cyclic polling systems to the waiting time in the random polling systems we realize that for both systems where the comparison is applicable (exhaustive and gated) the delay of the random system is greater by a term of:

$$\frac{(N-1)r}{2(1-N\mu)}$$

This observation is quite intuitive since we expect the delay in the random polling system to be higher due to the random behavior of the server. Note, also, that when the number of stations is $N=1$, then the delay in the random system is identical to the delay in the cyclic system.

Comparing the exhaustive service to the gated service (in both types of polling methods) we realize that the expected delay in the gated system is higher. Again, the difference in performance between the exhaustive scheme to the gated scheme is the same for both types of polling methods.

Looking at stability conditions, we see that both the gated system and the exhaustive system are stable (under both types of polling methods) as long as $N\mu<1$. On the other hand, the non-exhaustive scheme is stable only as long as $N\mu(1+r)<1$. This result is also intuitive since at least one switch over period (whose expected length is r) is associated with every

Polling Method / Service Method	Cyclic	Random
Exhaustive	$\frac{\delta^2}{2r} + \frac{\sigma^2}{2\mu(1-N\mu)} + \frac{Nr(1-\mu)}{2(1-N\mu)}$	$\frac{\delta^2}{2r} + \frac{\sigma^2}{2\mu(1-N\mu)} + \frac{Nr(1-\mu)}{2(1-N\mu)} + \frac{(N-1)r}{2(1-N\mu)}$
Gated	$\frac{\delta^2}{2r} + \frac{\sigma^2}{2\mu(1-N\mu)} + \frac{Nr(1+\mu)}{2(1-N\mu)}$	$\frac{\delta^2}{2r} + \frac{\sigma^2}{2\mu(1-N\mu)} + \frac{Nr(1+\mu)}{2(1-N\mu)} + \frac{(N-1)r}{2(1-N\mu)}$
Non Exhaustive	not available	$\frac{\delta^2}{2r} + \frac{\sigma^2}{2\mu(1-N\mu-Nr\mu)} + \frac{Nr\sigma^2}{2\mu(1-N\mu-Nr\mu)} + \frac{(N-1)r}{2(1-N\mu-Nr\mu)} + \frac{N\delta^2\mu}{2(1-N\mu-Nr\mu)}$

Table 4.1: the expected system time in the different systems

customer served.

CHAPTER 5

An Analysis of the Exhaustive Slotted ALOHA System

In this chapter we study the queueing behavior of exhaustive slotted ALOHA, a method which is used to control the transmission of N stations in a single channel radio environment. The exhaustive scheme can be considered as a simple method for synchronizing the transmission of the different stations; it gains its efficiency from this synchronization. Due to the synchronization the events in these system are strongly correlated to each other. For this reason, the system does not lend itself to a simple analysis and approximation methods must be used. It is observed that in certain cases that the exhaustive ALOHA system behaves similarly to the queue with starter system; in some other cases the system behaves like the random polling system. For this reason the results derived in chapters 2, 3 and 4 are used throughout this chapter.

5.1 Introduction and Previous Work

The (classic) ALOHA access scheme [Abra70] is a method which can be employed in shared channel networks (e.g. in radio networks or in satellite networks). In this access scheme a station transmits a packet as soon as the packet arrives, and packets transmitted concurrently collide and are not received at the receiving station. Collided packets are retransmitted after a randomly selected period of time, until they are successfully received.

The *slotted ALOHA* scheme [Robe72] is identical to the ALOHA scheme, but in addition, the time axis is divided to equal length slots (each of them equals in length the transmission time of a packet) and the stations are allowed to start transmitting only at the beginning of the time slots. Usually, the retransmission of collided packets in the slotted ALOHA scheme is done as follows: at every slot after the collision slot, a station i who participated in the collision, will retransmit the collided packet with probability p_i. Station i will continue this procedure until the collided packet is successfully received.

The ALOHA scheme has been extensively studied in the past. However, most of the past studies neglected the queueing behavior of the system by making simplistic assumptions either about the arrival process, or about the stations' buffer size. Naturally, the main result derived by these studies, is the throughput of the system as a function of the offered load [Abra70, Abra73, Robe72]. Several of these past studies (see [Lam74], for example) analyzed the delay in the slotted ALOHA system. However, in those studies the basic system model does not allow any queueing behavior, so the expressions derived in these reports mainly represent retransmission (of a packet) delay and not any queueing delay.

The queueing analysis of ALOHA has been done mainly for very simple systems. Sidi and Segall [Sidi83] analyzed the expected delay in an $N=2$ station slotted ALOHA system. This analysis is done under the assumptions that the arrival process to each station is a Bernoulli process and that the stations are symmetric. The expected delay in this system is found to be:

$$T = 1+\frac{(1-p)^2+ \cdot qp}{p(1-p)-q} \tag{5.1}$$

where p is the transmission probability used by each station to resolve* collisions, and q is the Bernoulli arrival rate to each queue. The *optimal* transmission probability, p^*, that minimizes this delay is:

$$p^* = 1-\frac{0.5q+[0.5q(1-q+0.5q^2)]}{1-0.5q} \tag{5.2}$$

The *exhaustive slotted ALOHA* scheme is a variation of the slotted ALOHA protocol. According to this scheme, a station whose packet is successfully received in slot t, is granted the exclusive right to transmit in the t+1st slot. By this mechanism, the exhaustive scheme overcomes one of the main deficiencies of the slotted ALOHA scheme: the continuous "fight" among the stations for every transmitted packet. Using this simple mechanism, the system takes advantage of the knowledge of the system history: the behavior of all stations during the t+1st slot depends on the channel status during the tth slot. The exhaustive scheme is called, sometimes, a *reservation scheme* or, R-ALOHA. It should be noticed that the exhaustive ALOHA scheme is based on the ability of the stations to recognize for each slot if the slot is a successful slot, a collision slot, or an idle slot.

It is obvious that the behavior and the advantages of the exhaustive scheme, cannot be studied without considering a queueing model of the system. Clearly, in a system where each of the stations is equipped with a single buffer queue (as assumed in many of the traditional models) the use of the exhaustive discipline will have no effect on the system performance.

For this reason, in our analysis we cannot use the simplistic model which neglects queueing behavior. To understand the system behavior, and to analyze its performance, it will be required, in this chapter, to use a queueing systems model.

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*According to this model in every time slot t, a station who has a packet to transmit will transmit with probability p.

The queueing behavior of the exhaustive ALOHA system has been studied before in two references. Lam [Lam80] studied the delay in the system by decoupling the behavior of each queue from the behavior of the rest of the system. The decoupling is done by assuming that the probability that a station successfully transmits in a contention slot equals to the average throughput of the (non exhaustive) slotted ALOHA system. This assumption leads to an approximation of the delay in the system; an approximation which is relatively good for large (around 40) number of stations (as shown in[Lam80]) but may not approximate the system properly when the number of stations is small. This analysis yields a small set of equations which must be solved by numerical methods to give the expected delay in the system.

In a recently published paper, Hofri and Konheim [Hofr84] give an exact analysis of the N-station exhaustive slotted ALOHA system. The analysis yields a set of several (relatively complex) equations that must be solved by numerical methods in order to derive the expected delay in the system.

In contrast to the previous studies the aim of the analysis done in this chapter, is to derive an *exact closed form expression* for the expected delay in a two-station exhaustive slotted ALOHA system; for the N-station system we derive a heavy-traffic closed form expression approximating the expected delay. In addition, in our analysis we will insist on deriving the expected delay not only for systems consisting of many stations (as did[Lam80]) but also for systems consisting of a few stations. The analysis of systems consisting of a few stations is important for the understanding of the behavior of multihop packet radio networks. In these networks, the local neighborhood of a given station does not necessarily consist of many stations. As a matter of fact, if routing algorithms are used, it is reasonable to believe that every local environment consists of a few stations which have many packets to transmit. This is in contrast to the model where in a one hop environment there are many stations, each of which transmits only a very few packets.

It is reasonable to believe that in a system consisting of only a few stations, the behavior of each station is strongly correlated to the behavior of the other stations. In contrast, in a system consisting of many stations, this correlation is much weaker. Thus, simplistic assumptions about independence of events, which are reasonable for many station systems, may be wrong for a system consisting of only a few stations. For this reason in the following analysis, we will avoid, as much as we can, the use of "independence" assumptions.

The analysis done in this chapter is divided into three main parts. First, in sections 5.3, 5.4, 5.5 and 5.6 we study the behavior of a *two-station* system. It is observed that an exhaustive slotted ALOHA system consisting of exactly two stations possesses special properties that make its behavior different from the behavior of the N-station system. This special behavior allows us to derive exact expressions for the expected delay in this system. In section 5.3 we present an approach for analyzing the expected delay in two-station exhaustive ALOHA systems. This approach analyzes the system by viewing it as *a queue with starter* system, and by using the results derived in chapters 2 and 3. In section 5.4 we analyze the expected delay in the *polite symmetric system* (to be defined below). In section 5.5 we analyze the expected delay in the *noisy system* (to be defined below); this analysis is done both for *symmetric* and *non symmetric* systems. The results derived in sections 5.4 and 5.5 are discussed in section 5.6.

The second part of this chapter, deals with the analysis of the expected delay in an *N-station* system; this analysis is done in section 5.7. Here we cannot use the method used in sections 5.4 and 5.5. Instead, we suggest a heavy load approximation of the expected delay. This approximation is achieved by viewing the system as an N-station random-polling exhaustive-service system and by using the results derived in chapter 4.

lastly, in section 5.8 we study a two-station system where the stations are not allowed to assume that they exist simply in a two-station system. In this case the special system behavior, observed in sections 5.3, 5.4 and 5.5 does not hold. To analyze the system, we use the results derived in sections 5.4 and 5.5 as a low load approximation and the results derived in section 5.7 as a heavy load approximation. These approximations are compared to simulation results, and shown to be good approximations.

5.2 Model Description

The system considered in this chapter consists of N stations (for some of the analysis we will assume N=2) which transmit packets of fixed length to each other. The transmission medium considered is a single radio channel. In this medium, if the transmissions of two (or more) stations overlap all transmitted packets get garbled, and we say that the packets *collided* (or, a *collision* occurred). Therefore, a transmission of a packet from station i to station j is successful if and only if no other station's transmission overlaps it.

The model considered is a discrete time model: time is slotted with the slot size equal to the service time of a fixed length packet, which, without loss of generality, we assume to be unity. The time interval $(t-1,t)$ is called the tth slot. It is assumed that a packet which arrives during the tth slot may be served in the tth slot, i.e., it is assumed that packets arrive at the beginning of the slot (this is called *the early arrival model [Klei64]*). A slot in which exactly one station transmits is called a *successful slot*. A slot in which two or more stations transmit is called a *collision slot* (or, a *conflict)*. A slot in which no station transmits is called an *idle slot*. It is assumed that each of the stations can recognize for each slot if the slot is a successful slot, a collision slot or an idle slot.

The arrival process to station i ($i=1,2,\cdots,N$) is an independent Bernoulli process with parameter q_i. Thus, the number of packets arriving at station i at slot t is given by:

$$\Pr[\ \#\text{ of packets arriving at station } i \text{ at slot } t = k] = \begin{cases} 1-q_i & k=0 \\ q_i & k=1 \\ 0 & k>1 \end{cases}$$

Packets which arrive to a given station, are transmitted from this station according to a first-come first-served order. Each station is equipped with an *infinite buffer* queue to store the packets which have not been transmitted yet.

The access scheme studied in this chapter is the exhaustive slotted ALOHA scheme; this scheme will be compared to slotted ALOHA which we must first describe. *Slotted ALOHA* is a method for resolving conflicts by probabilistic means. According to this method, a station, say i, which is involved in a collision (namely, transmitted a packet which collided with another packet) will continue trying to transmit the packet. To avoid an eternal deadlock, this station will not transmit in any slot succeeding the collision slot. Rather, in any of the succeeding slots the station will transmit with probability p_i and stay quiet with probability $1-p_i$. The station will continue this collision resolution scheme until its packet gets successfully transmitted. p_i, the probability for transmission of a packet in any slot succeeding a collision, is called the *transmission probability*. We assume that the transmission propagation is instantaneous. Thus, at the end of the tth slot, all stations know what was the status of the channel (successful transmission, idle or collision) in the tth slot. In addition, it is assumed that the channel is error free and instantaneous and free acknowledgements are assumed.

The *exhaustive slotted ALOHA* protocol (which is sometimes called *R-ALOHA)* is the same as slotted ALOHA scheme with an additional property. According to this method, a station, say i, who successfully transmits in the tth slot, is granted the exclusive right to transmit in the $t+1$st slot. This means that all other stations stay silent in the $t+1$st slot, letting station i transmit with no interruption. Therefore, according to this scheme, once station i has successfully transmitted a packet it continues transmitting until its buffer is completely *exhausted.*

It is obvious that in this protocol some additional means are required to signal the other stations when station i finishes transmitting. One way to indicate an end of transmission is just by not transmitting. The idle slot succeeding the transmission period of station i signals the other stations that station i has no more packets to send. Another method, and this is the method considered in this analysis, is to use an *end-of-use* flag. According to this method, the last bit of every packet sent by station i is devoted to notifying the other stations if station i intends to use the channel on the next slot.

Three types of periods, classified according to the channel activity, are distinguished in this system:

1. *Transmission period:* this period consists of a sequence of successful slots, all of them from the same station. The last packet in such a sequence has the end-of-use flag turned on.

2. *Idle period:* a period in which none of the stations has any packet to transmit, and the channel is idle.

3. *Contention period:* During this period there exist some (at least one) stations who wish to transmit, but none of the slots, due to collisions, is successful. Due to the probabilistic behavior of the ALOHA scheme, it is clear that some of the slots in a contention period may be idle (all the stations involved in the conflict stay quiet) while the other slots in the period must be collisions.

From this classification we conclude that:

1. A transmission period may be followed either by another transmission period (if station i transmits end-of-use flag in the tth slot and some station j successfully transmits in the $t+1$st slot), or by an idle period (empty system), or by a contention period.

2. An idle period may be followed either by a transmission period or by a contention period.

3. A contention period is always followed by a transmission period.

Exhaustive slotted ALOHA schemes may differ from each other by the specific conflict resolution protocol used during the contention period. In the following we consider two basic schemes:

1. *Polite ALOHA conflict resolution protocol* with transmission probabilities $p_1, p_2, \cdots, p_N$. According to this method, if station i wants to transmit, and if the channel is not currently exhausting any other station's buffer (namely, this is not a transmission period), then station i $(i=1,2,\cdots,N)$ will transmit with probability p_i.

2. *Noisy ALOHA conflict resolution protocol* transmission probabilities $p_1, p_2, \cdots, p_N$. According to this method, a station, say i, who wishes to transmit following an idle period or following a transmission period, will transmit with probability 1. If this transmission is successful, then station i is already in a transmission period. Otherwise, station i collided with other stations, and now it will transmit, in every slot, with probability p_i. The ith station will continue to transmit with probability p_i until one of the competing stations wins the conflict and starts a transmission period. At the end of this

transmission period, all* the stations who wish to transmit will again become "noisy" (transmit with probability 1) and then the algorithm repeats.

5.3 An Approach for the Two Station Case: The Queue with Starter

An exhaustive ALOHA system which consists of exactly two stations possesses a very important property: When station i ($i=1,2$) stops transmitting, declaring, by the end-of-use flag that it has no more packets to send, then the other station can use the next slot without any interruption. Thus, in a two-station exhaustive ALOHA scheme, a transmission period of station 1 will be followed by a transmission period of station 2 (unless station 2 has nothing to transmit when the transmission period of station 1 ends) and vice versa. Therefore, we can conclude that in this system, once a transmission period starts, a successful transmission will be seen on the channel until the whole system (namely both buffers) empties. Thus a contention period in the 2 station system will occur only after an idle period, which is not the case in a N-station system.

A natural approach to analyze this system is as follows: Derive the number of packets found at queue i at an arbitrary point of time and then, from this result, derive the waiting time at that station. However, this approach is not trivial since the behavior of one station depends on the state of the other station.

For this reason we choose an alternative approach to calculate the expected delay. We consider the whole system as a black box and derive the expected waiting time of the total system. We assume that the arrival stream entering the *black box system* is identical to the combined arrival stream entering the real system. The service time, in the black box system is assumed to be, as in the real system, one unit of time. The server behavior, in the black box system, is identical to the server behavior in the real system, according to the following rule: A customer is served in the black box system *if and only if* a packet is successfully transmitted in the real system. It is clear, under this "simulation" rule, that the number of customers found at time t in the black box system must be identical to the total number of packets found at time t in the real system. Figure 5.1 depicts the relation between the real system and the black box system.

It is true that if the black box system serves the customers in a first-come first-served fashion, and if each station in the real system serves its customers in the same manner, then the actual delays observed in the two systems will not be identical. Nevertheless, it is obvious that, at every moment, the total number of customers (packets) found in both systems is the same, so the *expected number* of customers found in both systems is the same. Therefore, using the black

*A station whose buffer is empty at the beginning of the contention period (i.e., a station who does not transmit in the "noisy slot") and which generates an arrival during the contention period, will transmit only at the end of the following transmission period.

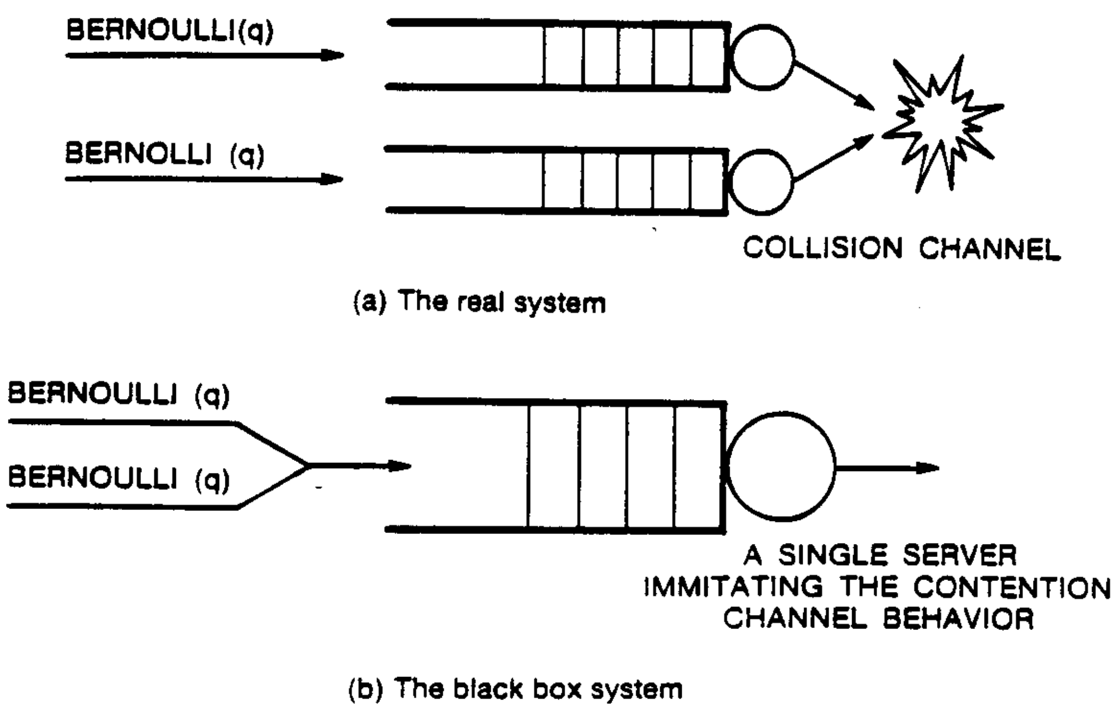

Figure 5.1: The two-station ALOHA system and the corresponding black box system

box system we can easily find the expected (total) number of packets found in the real system.

Moreover, since the number of customers arriving to the black box system at slot t equals the total number of packets entering the two-station ALOHA system, it is obvious that the arrival rate in the two systems is identical. Thus, using Little's result we can conclude that the expected system time (waiting time plus service time) of an arbitrary customer in the black box system is identical to the expected system time of an arbitrary packet in the real system. The use of Little's result in this case is allowed, since this result is valid for any work-conserving system, independent of the arrival process, and independent of the service policy used in the system.

Let us start by analyzing the black box system. From the behavior of the real system, as described above, the black box system will behave as follows:

1. Once the server in the black box system starts serving customers, it will continue serving, at a normal rate of one customer per one unit of time, until the queue of this system

becomes empty. This behavior actually corresponds to the sequence of alternating transmission periods in the real system.

2. When the black box system becomes empty the server will stay idle as long as no customers arrive to this system. This period is the *idle period* of the black box system, and it corresponds to the idle period of the real system.

3. When new customers arrive to the black box system, following an idle period, they will not be served immediately. Rather, some time will elapse before the first of these customers will be served. This period of time, starting at the first arrival to an idle system, and ending at the beginning of the first service slot (following the idle period) corresponds exactly to the contention period in the real system. This is a period of time where there are packets in the real system but no packet is transmitted (a period of time where there are customers in the black box system but no customer is served). We call this period, in the black box system, a *warm up period* (i.e., the cold start period) since at this time the server can be thought as "warming up" after being idle and before getting ready to serve the new arriving customers.

From the above description, one can easily recognize that the black box system behaves exactly like the queue with starter, which was analyzed in chapters 2 and 3. Thus, we will analyze this system by using the results reported in the analysis of the queue with starter.

From the analysis done in chapters 2 and 3 let us recall that the expected delay in a queue with starter system can be calculated by going through the following steps:

i. Calculate the expected delay in the equivalent queue *without starter.*

ii. Calculate the first two moments of the start up (warm up) period.

iii. From (i) and (ii) above and the relations given in chapters 2 and 3 calculate the expected delay in the queue with starter.

In the following we will use this approach to derive the expected delay in two different systems:

a. A black box system which corresponds to a real system where the conflict resolution protocol is a *polite* exhaustive slotted ALOHA. This approach will allow us to solve only for the case where $p_1=p_2=1/2$ (section 5.4); for general p we will later obtain a solution using a Markov chain approach.

b. A black box system which corresponds to a real system where the conflict resolution protocol is a noisy slotted ALOHA (section 5.5).

5.4 Two-Station Polite Exhaustive ALOHA

In this section we study the expected delay in a symmetric two-station polite exhaustive ALOHA system. The system is analyzed under the assumption that the stations are fully symmetric; Thus the arrival rate for each station is q, and the transmission probability used by each station is p. Two approaches are used to solve this system: First we use the queue with starter approach, which allows us to solve for the expected delay in the system when $p=1/2$. Second, a Markov chain approach is used to derive the expected delay for an arbitrary p. The expressions derived by these two methods are compared to each other and shown to be equivalent.

The importance of using the queue with starter approach to analyze the system when $p_1=p_2=1/2$ is twofold: 1) The transmission probabilities $p_1=p_2=1/2$ are known to be the optimal transmission probabilities of a two-station ALOHA system. These are optimal in a model where queueing is not modeled and the performance criteria is to maximize the system throughput. 2) As we see in the following, this specific system under the assumption that $p_1=p_2=1/2$ easily lends itself to analysis, by using the queue with starter approach. Thus, the (relatively simple) results derived in this analysis can be used later to verify results derived in the analysis of the more general system.

5.4.1 A Special Case Using the Queue with Starter Approach

In the following we analyze the expected delay in an exhaustive ALOHA system where the conflicts are resolved by the polite ALOHA conflict resolution scheme, with parameters $p_1=p_2=1/2$. As suggested above, the system will be studied by analyzing the black box system and by using the discrete-time queue with starter results.

The queue with starter system to be studied is the following:

a. The model is a discrete time model, measured in units of slots.

b. Arrivals consist of the merged stream of two Bernoulli sources, each with parameter q.

c. Service times are equal to one slot.

d. Cold start periods, correspond to the contention periods in the exhaustive slotted ALOHA system.

Let us recall that that the delay in a queue with starter can be calculated as a sum of two random variables:

- the *original delay*, which is the delay suffered in the queue without starter.

- the *additional delay,* an additional delay which is due to the presence of a starter.

The first step in analyzing the queue with starter is finding the delay (system time) in the equivalent queue *without* starter, i.e., the system time in the equivalent system that does not suffer cold starts. This is a simple single server queue with arrivals as in the queue with starter, and regular service policies (the server serves a customer whenever there is some customer in the system). Let us analyze a discrete time single server queue where service times are one slot, and the number of customers arriving in any slot is taken from an arbitrary distribution. Let $L(t)$ denote the number of customers found in this system at time t (i.e. at the slot boundary) and let $M(z)$ denote the z-transform of $L(t)$ when the system is in equilibrium, i.e.:

$$M(z) \triangleq \lim_{t\to\infty} E[z^{L(t)}]$$

Let $X(t)$ be the number of the customers arriving to the system at slot (time) t and let:

$$V(z) \triangleq E[z^{X(t)}] \quad , \qquad \mu = E[X(t)] \quad , \qquad \sigma^2 = Var[X(t)]$$

From the system description we can relate the number of customers found in the system at time $t+1$ to the number of customers found in the system at time t as follows:

$$L(t+1) = \begin{cases} L(t)+X(t+1)-1 & \text{if } L(t)>0 \\ L(t)+X(t+1) & \text{if } L(t)=0 \end{cases} \tag{5.3}$$

From (5.3), in equilibrium we get:

$$M(z) = \frac{1}{z}\Big(M(z)-M(0)\Big)V(z) + M(0)\cdot V(z)$$

or:

$$M(z) = M(0)\cdot\frac{V(z)\cdot(z-1)}{z-V(z)} \tag{5.4}$$

Applying $M(1)=1$, and using L'Hospital's rule in (5.4), we get:

$$M(0) = 1-\mu$$

Now, differentiating (5.4) with respect to z and evaluating the derivative at $z=1$ we get the the known result of the expected queue length in the system (at the slot boundaries):

$$E[L] = \frac{dM(z)}{dz}\bigg|_{z=1} = \frac{\sigma^2+\mu-\mu^2}{2(1-\mu)} \tag{5.5}$$

Now, after analyzing a general-arrival system, recall that the arrivals to our system consist of the merged stream of two independent Bernoulli sources. Thus we have:

$$Pr[X(t)=k] = \begin{cases} (1-q)^2 & k=0 \\ 2q(1-q) & k=1 \\ q^2 & k=2 \\ 0 & \text{else} \end{cases}$$

and

$$\mu = E[X(t)] = 2q$$

$$\sigma^2 = Var[X(t)] = 2q(1-q)$$

$$V(0) = Pr[X(t)=0] = (1-q)^2$$

So, substituting μ and σ^2 into (5.5) we get:

$$E[L] = 2q + \frac{q^2}{1-2q} \tag{5.6}$$

Now, to find the expected delay in this system we apply Little's result:

$$E[\text{ system time in the " queue without starter" }] = \frac{E[L]}{2q} = 1 + \frac{q}{2(1-2q)} \tag{5.7}$$

Next, the second step in calculating the delay in the queue with starter, is to derive the additional delay suffered in the system. Let S denote the length of a cold start and let:

$$s_i \triangleq Pr[S=i] \ , \quad S(z) \triangleq \sum_{i=0}^{\infty} s_i z^i \ , \quad \overline{S} \triangleq \sum_{i=0}^{\infty} i \cdot s_i \ , \quad \overline{S^2} \triangleq \sum_{i=0}^{\infty} i^2 \cdot s_i$$

In addition, let $\overline{G}$ denote the expected value of the additional delay suffered by an arbitrary customer.

To calculate the additional delay suffered in the queue with starter one has to calculate the first and the second moments of the cold start. We recall that the cold start in this system corresponds to the contention period in the exhaustive slotted ALOHA system. The length of this contention period is the number of slots from the arrival of a customer to the empty system, until a first packet is successfully transmitted (not including the successful slot). Let us find the probability that a slot, t, in the contention period will be a successful transmission. Clearly, this probability depends on the number of stations who wish to transmit at time t. There exist two cases:

1. Only one station wishes to transmit. In this case the probability of having a successful transmission at time t is p, and since we have $p=1/2$, the probability of a successful

transmission is 1/2.

2. Both stations wish to transmit (i.e., both buffers are not empty). In this case the probability of a successful transmission is $2p(1-p)$, which again, under the assumption that $p=1/2$, equals to 1/2.

From these properties it is easy to realize that independently of the number of stations wishing to transmit, the length of the cold start is distributed as:

$$s_i = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^i \quad ; \qquad i=0,1,2,\cdots \tag{5.8}$$

It is important to emphasize that the length of the cold start, in this carefully selected case, is *independent* of the number of customers arriving to the system during the contention period (note that for $p \neq 1/2$ this property fails and thus the queue with starter approach cannot be used to analyze the general system). Therefore, we are allowed to apply the results derived in chapter 2 (the analysis of a queue with an independent starter). From (5.8) it is easy to derive the z-transform and the first two moments of the cold start length:

$$S(z) = \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} \ , \qquad \overline{S} = 1 \ , \qquad \overline{S^2} = 3 \tag{5.9}$$

Now that we have the first two moments of the the length of the cold start we recall that the expected value of the additional delay suffered in the queue with starter can be calculated from these two moments and from $V(0)$ (the probability that no customer arrives to the system at a given slot, t). This is done using the following expression (taken from chapter 2):

$$\overline{G} = \frac{2\overline{S}+(\overline{S^2}-\overline{S})(1-V(0))}{2+2\overline{S}(1-V(0))}$$

Substituting $\overline{S}$, $\overline{S^2}$ and $V(0)$ into this equation we get the expected value of the additional delay in the system:

$$\overline{G} = \frac{2 + 2(1-V(0))}{2 + 2(1-V(0))} = 1 \tag{5.10}$$

Note, that the additional delay in this system is always 1, independent of the arrival rate. This property is not surprising, due to the fact that the cold start of this system is geometrically distributed, and since the geometric distribution was shown (in chapter 2) to be the eigenfunction of the queue with starter system.

Now, after calculating the original delay in the queue without starter, and calculating the additional delay in the queue with starter, we can add them up and finally get the total delay (system time) in our black box system:

$$T = E[\text{ system time in the queue without starter }] + \overline{G} = 2 + \frac{q}{2(1-2q)} \qquad (5.11)$$

Thus, we conclude that the expected system time of an arbitrary packet in the two-station polite exhaustive slotted ALOHA system (with parameters $p_1 = p_2 = 1/2$) is:

$$T = 2 + \frac{q}{2(1-2q)}$$

5.4.2 The General Symmetric Case: A Markov Chain Approach

In section 5.4.1 we used the queue with starter approach to derive the expected delay in a two-station polite ALOHA system where the transmission probability of the stations was $p_1 = p_2 = 1/2$. The goal of this section is to calculate the expected delay in this system for an arbitrary (yet symmetric) transmission probability (i.e., $p_1 = p_2 = p$).

The analysis method to be used in this section is a direct one: we construct the Markov chain representing the number of packets in the system and derive from this chain the expected number of packets and the expected delay in the system.

As in previous sections, due to symmetry in the system, our interest is in the *total* number of packets found in the system. For this reason, a state in the Markov chain should represent the total number of packets in the system. However, this representation is not sufficient; it is also required to represent the specific state of the system. These states can be any one of four types:

1. The system is in idle period. This state is called *an idle state.*

2. The system is in a contention period, and only one queue contains packets (the other queue is empty). These states are called *single-contention states.*

3. The system is in a contention period and both queues have packets. These states are called *double-contention states.*

4. The system is in exhaustive transmission period. These states are called *An exhaustive*

transmission states.

It is easy to see that if we define the system state to be specified by the specific state type (one of the four mentioned above) and by the number of packets in the system, we have then formed a Markov chain.

Based on these definitions we denote the following:

$$c_0 \triangleq Pr[\text{ system is in the idle state }]$$

$$c_i \triangleq Pr[\text{ system is in a single-contention state }, i \text{ packets in system }] \quad ; \quad i=1,2,\cdots$$

$$d_i \triangleq Pr[\text{ system is in a double-contention state }, i \text{ packets in system }] \quad ; \quad i=0,1,2,\cdots$$

$$e_i \triangleq Pr[\text{ system is in an exhaustive transmission state }, i \text{ packets in system }] \quad ; \quad i=0,1,2,\cdots$$

It is clear from the state definition that $d_0=0$, $d_1=0$, and $e_0=0$.

In addition to these definitions, m_i denotes the probability that there are i packets in the system, independently of the system state:

$$m_i = Pr[\text{ there are } i \text{ packets in the system }]$$

so we have:

$$m_i = c_i + d_i + e_i \quad ; \qquad i=0,1,2,\cdots$$

Based on this notation, let us define the z-transforms of the number of packets found in the system:

$$C(z) = \sum_{i=0}^{\infty} c_i z^i$$

$$D(z) \triangleq \sum_{i=2}^{\infty} d_i z^i$$

$$E(z) \triangleq \sum_{i=1}^{\infty} e_i z^i$$

$$M(z) \triangleq \sum_{i=0}^{\infty} m_i z^i$$

In addition we give notation for the probability of the events which can occur in the system:

$a_0 \triangleq Pr[$ no packet arrives to the system at time $t]$

$a_1 \triangleq Pr[$ one packet arrives to the system at time $t]$

$a_1^1 \triangleq Pr[$ a packet arrives to queue 1, no packet arrives to queue 2, at time $t]$

$a_1^2 \triangleq Pr[$ a packet arrives to queue 2, no packet arrives to queue 1, at time $t]$

$s_1 \triangleq Pr[$ successful transmission at potential* contention slot $|$ one queue is empty and the other is not $]$

$f_1 \triangleq Pr[$ no successful transmission at potential contention slot $|$ one queue is empty and the other is not $]$

$s_2 \triangleq Pr[$ successful transmission at potential contention slot $|$ both queues are not empty $]$

$f_2 \triangleq Pr[$ no successful transmission at potential contention slot $|$ both queues are not empty $]$

Let us recall that the arrival process to each station is an independent Bernoulli process with parameter q, and that a busy station transmits at every contention slot with probability p. Thus, from the notation defined above we have:

$$a_0 = (1-q)^2 \ , \qquad a_1 = 2q(1-q) \ , \qquad a_2 = q^2$$

$$a_1^1 + a_1^2 = a_1 \quad \text{and due to symmetry:} \quad a_1^1 = a_1^2 = \frac{a_1}{2} = q(1-q)$$

$$s_1 = p \ , \qquad f_1 = 1 - s_1 = 1 - p$$

*A potential contention slot is a non-idle slot which does not follow a transmission slot. In a potential contention slot some of the stations are willing to transmit. If only one station transmits in this slot, this is a transmission slot, otherwise it is a contention slot.

$$s_2 = 2p(1-p) \ , \qquad f_2 = 1-s_2 = 1-2p(1-p)$$

Figure 5.2 depicts this Markov chain. A state in this figure is denoted by a couple: the first element of the couple denotes the system state type (I denotes idle, C_1 denotes single-contention, C_2 denotes double-contention and E denotes exhaustive transmission). The second term in the couple denotes the number of packets in the system. Three main rows can be observed in this figure. The upper row in the chain (figure 5.2a) contains the double-contention states. The middle row contains both the idle state and the single-contention states. The lower row contains the exhaustive transmission states.

Due to the complexity of the figure we represent some of the state transitions in figure 5.2a and the others in figure 5.2b. Figure 5.2a depicts all the states and the rows as described above. In addition it depicts the transitions entering states in the double-contention row and transitions entering states in the single-contention and idle row. Figure 5.2b depicts all the transitions entering states in the exhaustive transmission row; for the sake of clarity the double-contention row (which is the upper row according to figure 5.2a) is shifted to the bottom of the figure.

Using the notation defined above, our next goal is to derive the z-transform of the number of packets in the system. First, we deal with the single-contention states, represented by the middle row in figure (5.2a). From this figure we get the following equilibrium relations:

$$c_1 = c_0 a_1 + c_1 a_0 f_1 \tag{5.12a}$$

$$c_i = c_{i-1} a_1^1 f_1 + c_i a_0 f_1 \quad ; \quad i=2,3,\cdots \tag{5.12b}$$

From these relations we solve for c_i:

$$c_i = \left(\frac{c_0 a_1}{1-a_0 f_1}\right) \cdot \left(\frac{a_1^1 f_1}{1-a_0 f_1}\right)^i \quad ; \quad i=1,2,\cdots \tag{5.13}$$

From equation (5.13) we find $C(z)$ as follows:

$$C(z) = \sum_{i=0}^{\infty} c_i z^i = c_0 + \frac{c_0 a_1 z}{1-a_0 f_1 - a_1^1 f_1 z} \tag{5.14}$$

Next, we deal with the double-contention states. From the Markov row we get:

$$d_2 = c_0 a_2 + c_1 a_1^2 f_1 + d_2 a_0 f_2 \tag{5.15a}$$

$$d_i = c_{i-2} a_2 f_1 + c_{i-1} a_1^2 f_1 + d_{i-1} a_1 f_2 + d_i a_0 f_2 \quad ; \quad i=3,4,\cdots \tag{5.15b}$$

From these equations we get $D(z)$ expressed in terms of $C(z)$:

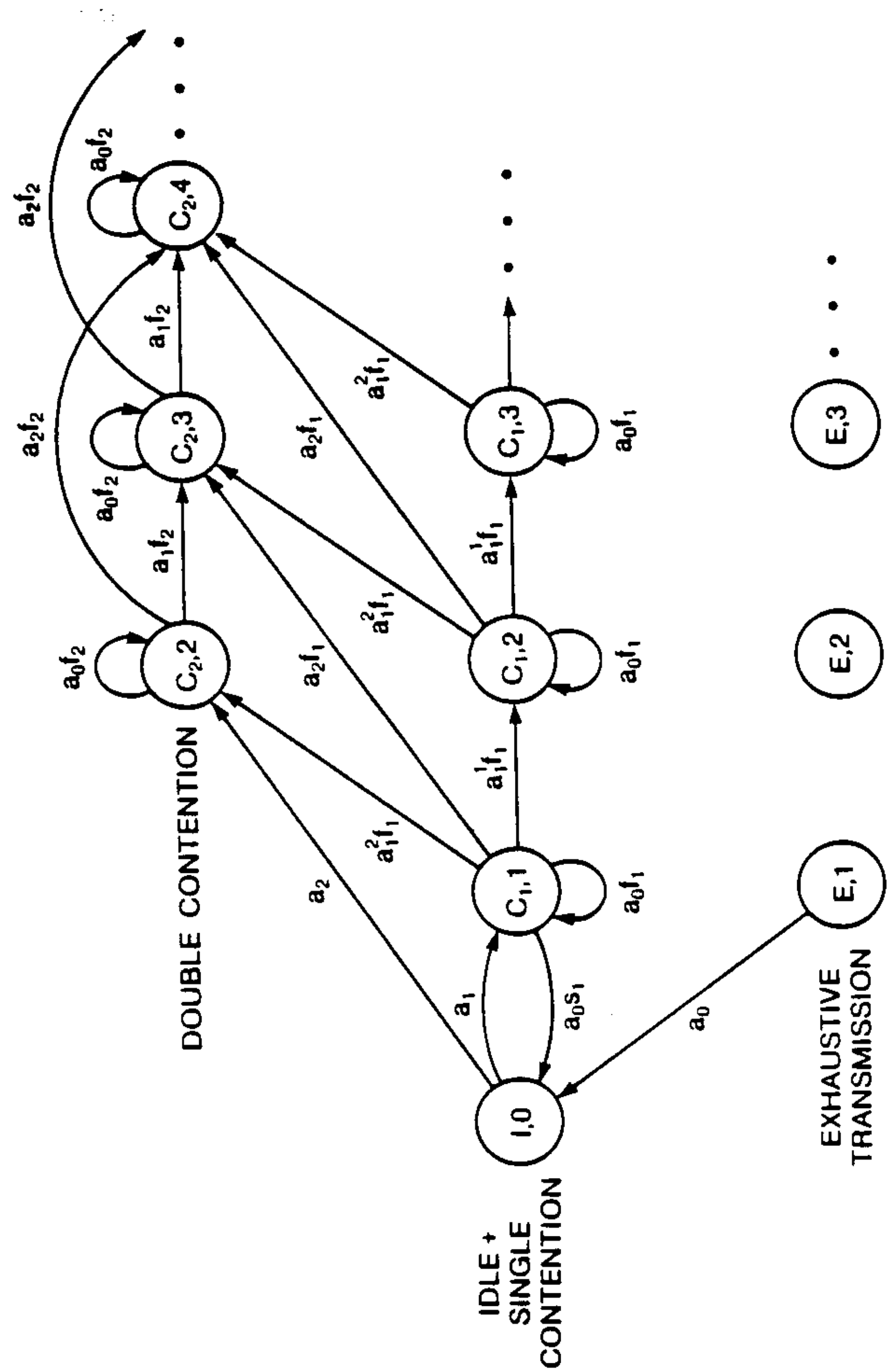

Figure 5.2a: The state diagram of the number of packets in the system

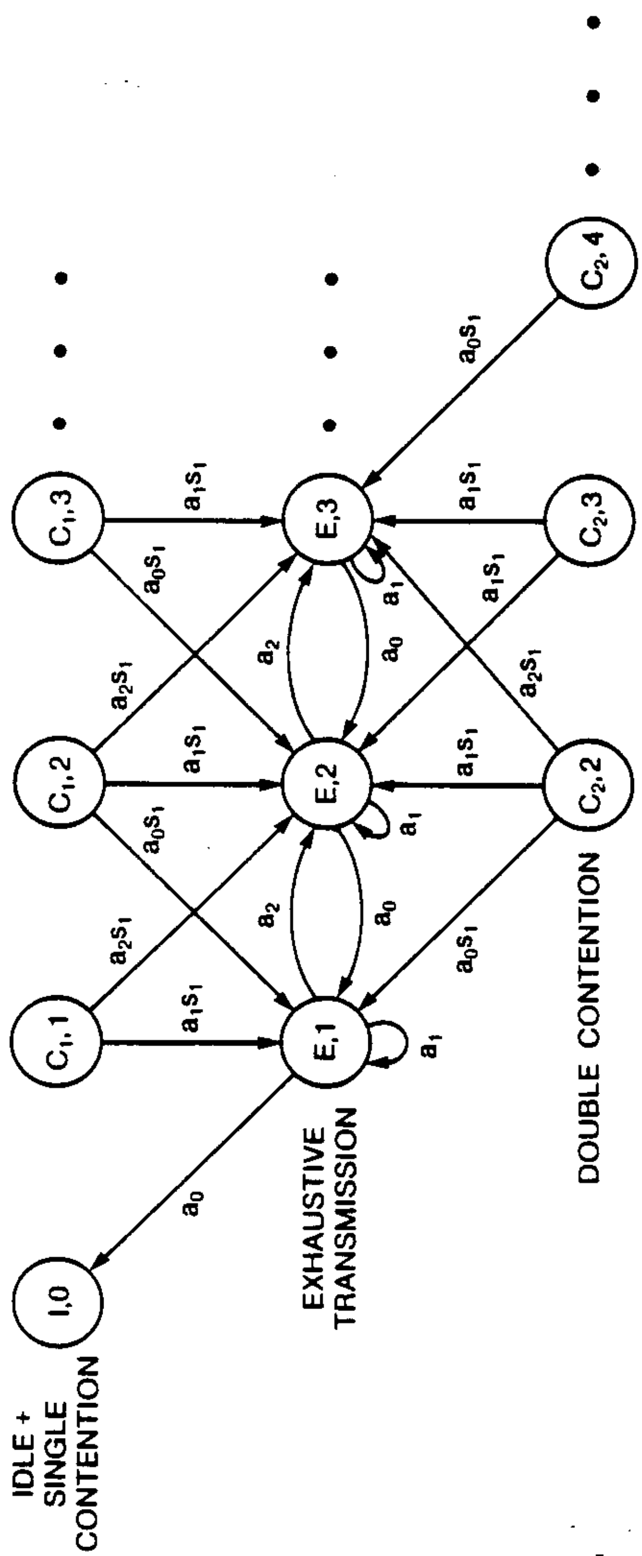

Figure 5.2b: The state diagram of the number of packets in the system

$$C(z) = \frac{C(z)\cdot[a_2 f_1 z^2 + a_1^2 f_1 z] \;+\; c_0[a_2(1-f_1)z^2 - a_1^2 f_1 z]}{1 - a_2 f_2 z^2 - a_1 f_2 z - a_0 f_2} \tag{5.16}$$

Thus, the z-transform of the top row can be expressed in terms of the z-transform of the middle row.

Last, we deal with the bottom row, representing the exhaustive transmission states. For these states we have:

$$e_1 = (c_1 a_1 + c_2 a_0)\cdot s_1 + (d_1 a_1 + d_2 a_0)\cdot s_2 + (e_1 a_1 + e_2 a_0) \tag{5.17a}$$

$$e_i = (c_{i-1} a_2 + c_i a_1 + c_{i+1} a_0)\cdot s_1 \;+\; (d_{i-1} a_2 + d_i a_1 + d_{i+1} a_0)\cdot s_2$$

$$+\; (e_{i-1} a_2 + e_i a_1 + e_{i+1} a_0) \qquad ; \qquad i=2,3,\cdots \tag{5.17b}$$

Now taking the z-transform of these equations we get:

$$E(z) = [s_1 C(z) + s_2 D(z) + E(z)]\cdot[a_1 + \frac{a_0}{z} + a_2 z] \;-\; a_1 c_0 - a_0 c_1 + \frac{a_0 c_0}{z} - c_0 a_2 z \;-\; \frac{a_0 e_1}{z} \tag{5.18}$$

Next, we use the equilibrium equation for the state $(I,0)$. This is:

$$c_0 = c_0 a_0 + c_1 a_0 s_1 + e_1 a_0 \tag{5.19}$$

Equation (5.19) is now used to calculate e_1:

$$e_1 = \frac{c_0}{a_0} - c_1 s_1 - c_0 \tag{5.20}$$

Now substituting equation (5.20) into (5.18) and manipulating (5.18) further gives:

$$E(z) = z\cdot\frac{s_1[P(z)-c_0] + s_2 D(z) - c_0(1-a_0)}{z - a_0 - a_1 z - a_2 z^2} \;-\; s_1[P(z)-c_0] \;-\; s_2 D(z) \tag{5.21}$$

From equations (5.14), (5.16) and (5.21) it is now possible to calculate the z-transform of the number of packets found in the system, in terms of c_0 and the system parameters. This can be done by:

$$M(z) = C(z) + D(z) + E(z) \tag{5.22}$$

From equation (5.22) we next can solve for c_0. This can be done by using the fact that the sum of the m_i's must yield 1. Thus, we calculate c_0 by solving the equation:

$$M(1) = 1 \tag{5.23}$$

Due to the complex structure of this Markov chain, this job (of solving equation (5.23) for c_0) is quite complicated. Thus, we are aided by a computer program* to derive this solution:

$$c_0 = \frac{(a_2-a_0)(a_2f_1-a_0f_1-f_1+2)(1-f_2)}{a_0a_2f_1f_2-a_0^2f_1f_2-a_0f_1f_2-2a_2f_2+2f_2+a_2f_1+3a_0f_1-f_1-2} \tag{5.24}$$

Now, substituting the values of a_0, a_2, f_1 and f_2 into equation (5.24) we get:

$$c_0 = \frac{4(p-1)p(2q-1)(pq-q-p)}{-2(2p^3q^3-4p^2q^3+3pq^3-q^3-6p^3q^2+14p^2q^2-9pq^2+2q^2+6p^3q-12p^2q+6pq-2p^3+2p^2)} \tag{5.25}$$

Let us evaluate c_0 for some known cases:

i. For $q=1/2$ and arbitrary p, we get $c_0=0$. This means that when the system is heavily loaded (total arrival rate to the system is one packet per one unit of time) then the probability that the system is empty, is zero.

ii. For $q=0$ and arbitrary p we get $c_0=1$. This means that when the arrival rate is zero, the system must be empty.

iii. For $p=0$ and $p=1$ and for an arbitrary value of q ($q\neq 0$) we have $c_0=0$. This is a case where the contention period could become infinitely long, either due to too low a transmission probability, or due to too high a transmission probability. In any of these cases, the probability that the system is empty, must be zero.

Now, that we have the value of c_0, we substitute it back into $M(z)$ (defined by equations (5.14), (5.16) and (5.21)). Next $M(z)$ is differentiated with respect to z, and the derivative is evaluated at $z=1$. This yields the expected value of the number of packets in the system. Again, due to the complexity of the z-transform, this task is quite complicated (even worse than finding c_0), and again we need to be aided by the computer to derive this expression. Once we have the expression for the expected number of packets in the system we use Little's result and divide this expression by the total arrival rate ($2q$) to yield the expected system time.

The expected system time is expressed in a rational form where both the numerator and the denominator are polynomials of p and q. Thus we have:

$$T = \frac{N(p,q)}{D(p,q)} \tag{5.26a}$$

where:

*Part of the algebraic manipulation required in this analysis was done by the computer program MACSYMA.

$$
\begin{aligned}
N(p,q) \;=\; & +2p^6q^5-8p^5q^5+17p^4q^5-23p^3q^5+19p^2q^5-9pq^5+2q^5 \\
& -8p^6q^4+40p^5q^4-88p^4q^4+108p^3q^4-77p^2q^4+30pq^4-5q^4 \\
& +12p^6q^3-72p^5q^3+161p^4q^3-179p^3q^3+102p^2q^3-28pq^3+2q^3 \\
& -8p^6q^2+60p^5q^2-134p^4q^2+132p^3q^2-57p^2q^2+8pq^2 \\
& +2p^6q-24p^5q+52p^4q-42p^3q+12p^2q \\
& +4p^5-8p^4+4p^3
\end{aligned} \tag{5.26b}
$$

and

$$
\begin{aligned}
D(p,q) \;=\; & 2(1-p)p(1-2q)(p+q-pq)\cdot \\
& [2p^3q^3-4p^2q^3+pq^3-q^3 \\
& -6p^3q^2+14p^2q^2-9pq^2+2q^2 \\
& +6p^3q-12p^2q+6pq \\
& -2p^3+2p^2]
\end{aligned} \tag{5.26c}
$$

The expression for the expected delay in the system, given in equation (5.26) is quite complex, and does not lend itself to an easy interpretation. For this reason, testing this expression for specific cases is essential for the understanding of the system behavior, and for the "verification" of this computer aided analysis.

When the arrival rate to the system is very low, i.e., when $q \to 0$ we have:

$$\lim_{q\to 0} N(p,q) = 4p^3(1-p^2)$$

and:

$$\lim_{q\to 0} D(p,q) = 4p^4(1-p^2)$$

and so we get:

$$\lim_{q\to 0} T = \frac{1}{p} \tag{5.27}$$

This behavior has a very simple explanation: When the load is very low, each packet arrives to an empty system and the delay it suffers is the time until it will be transmitted successfully.

Since no collision is likely to occur in the system (no other packets are likely to arrive to the system before it is transmitted) the delay is actually the time elapsing from the slot at which the packet arrives, until it is first transmitted. Since, in every slot following the arrival slot, the packet will be transmitted with probability p, the expected delay of the packet is $1/p$.

When the system is heavily loaded, i.e., when $q \rightarrow 1/2$, we have:

$$\lim_{q \rightarrow 1/2} N(p,q) = p(1-p)(1+p) \cdot \frac{3+9p-8p^2-2p^3}{32}$$

and:

$$\lim_{q \rightarrow 1/2} D(p,q) = \lim_{q \rightarrow 1/2} 2p(1-p)[\frac{1}{2}+\frac{p}{2}](1-2q) \cdot \frac{3+9p-8p^2-2p^3}{8}$$

and so we get:

$$\lim_{q \rightarrow 1/2} T = \lim_{q \rightarrow 1/2} \frac{1}{4(1-2q)} \tag{5.28}$$

Thus at heavy load, the delay in the system behaves as $\frac{1}{4(1-2q)}$. This is identical to the behavior of the queue without starter studied in section 5.4.1. Let us recall that the queue without starter studied in section 5.4.1 represents a single server queue where the arrival process is identical to the arrival process in our system. Now, the explanation of this result is trivial: When the exhaustive ALOHA system is heavily loaded, the system spends most of the time in the exhaustive transmission mode; in this mode of operation, the system behaves exactly like a single server queue (no conflicts, and perfect transmissions), so the expected delay approaches the expected delay of the single server queue.

When the transmission probability p is very small, i.e., when $p \rightarrow 0$, we have:

$$\lim_{p \rightarrow 0} N(p,q) = q^3(2-q)(1-2q)$$

$$\lim_{p \rightarrow 0} D(p,q) = \lim_{p \rightarrow 0} 2q^3(2-q)(1-2q) \cdot p$$

Thus we have:

$$\lim_{p \rightarrow 0} T = \lim_{p \rightarrow 0} \frac{1}{2p} = \infty \tag{5.29}$$

Again, this result is very intuitive: For a *fixed* arrival rate $0<q<1/2$ and when $p \rightarrow 0$ the major part of the delay is due to the contention period (the queueing effects are negligible compared to the contention period effect). In this case, since $p \rightarrow 0$ and the arrival rate is non-zero, both queues will be *non empty* during the major part of the contention period. Thus, the probability of a successful transmission during this period behaves like $2p(1-p)$ which approaches $2p$ when $p \rightarrow 0$. Thus the expected length of the contention period and the expected delay behave like $\frac{1}{2p}$.

When the transmission probability is very high, i.e., when $p \to 1$ we get a similar behavior:

$$\lim_{p \to 1} N(p,q) = q^2(1-2q)$$

and:

$$\lim_{p \to 1} D(p,q) = \lim_{p \to 1} 2q^2(1-2q)\cdot(1-p)$$

so:

$$\lim_{p \to 1} T = \lim_{p \to 1} \frac{1}{2(1-p)} = \infty \tag{5.30}$$

This behavior is symmetric to the behavior of the system when $p \to 0$, and the explanation is similar.

The last "verification" test, is to evaluate the system delay, when $p = 1/2$. We are able to perform this test since the behavior of the system, under this transmission policy, was independently studied in section 5.4.1. Evaluating the delay at $p = 1/2$ gives:

$$N(p=1/2,q) = \frac{(q+1)^2(7q-4)(q^2-2q-1)}{32}$$

and:

$$D(p=1/2,q) = \frac{(q+1)^2(2q-1)(q^2-2q-1)}{16}$$

so:

$$T(p=1/2,q) = \frac{4-7q}{2(1-2q)} = 2 + \frac{q}{2(1-2q)} \tag{5.31}$$

This expression, as expected, is identical to the expression (equation (5.11) we derived earlier by using the queue with starter method.

Next, we plot the expected delay in the system as function of the transmission probability (p) and the arrival rate (q) arriving to one of the queues. Figure 5.3 depicts the expected delay in the system as function of the transmission probability p. The figure contains several curves, each of them corresponds to different arrival rate (q). Figure 5.4 depicts the expected delay in the system as function of the single-station arrival rate q. The figure contains several curves, each of them corresponds to a different transmission probability (p). Figure 5.5 is a three dimensional plot, depicting the expected delay as function of both the transmission probability p and the single-station arrival rate q.

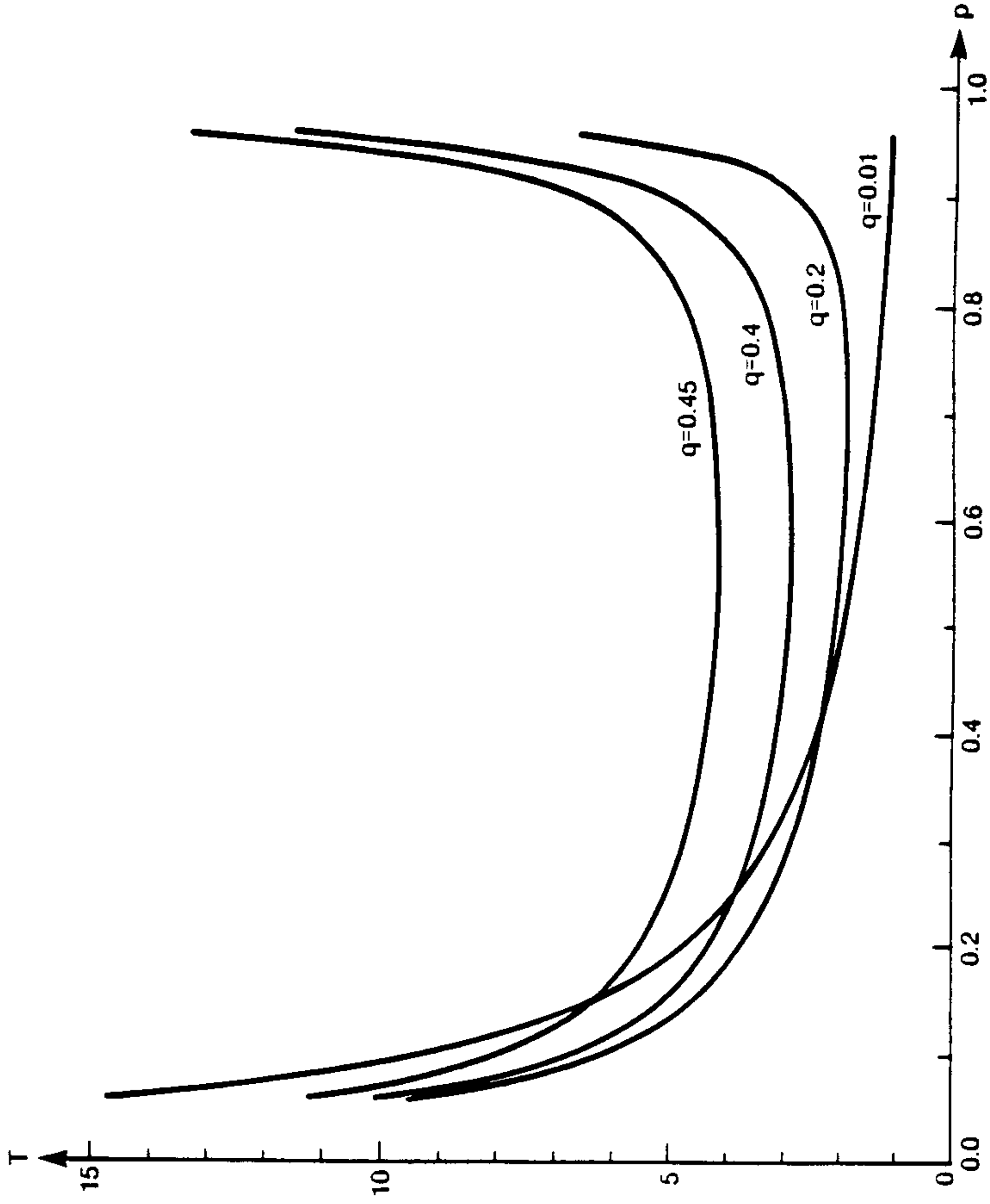

Figure 5.3: Polite system: The expected delay as function of the transmission probability

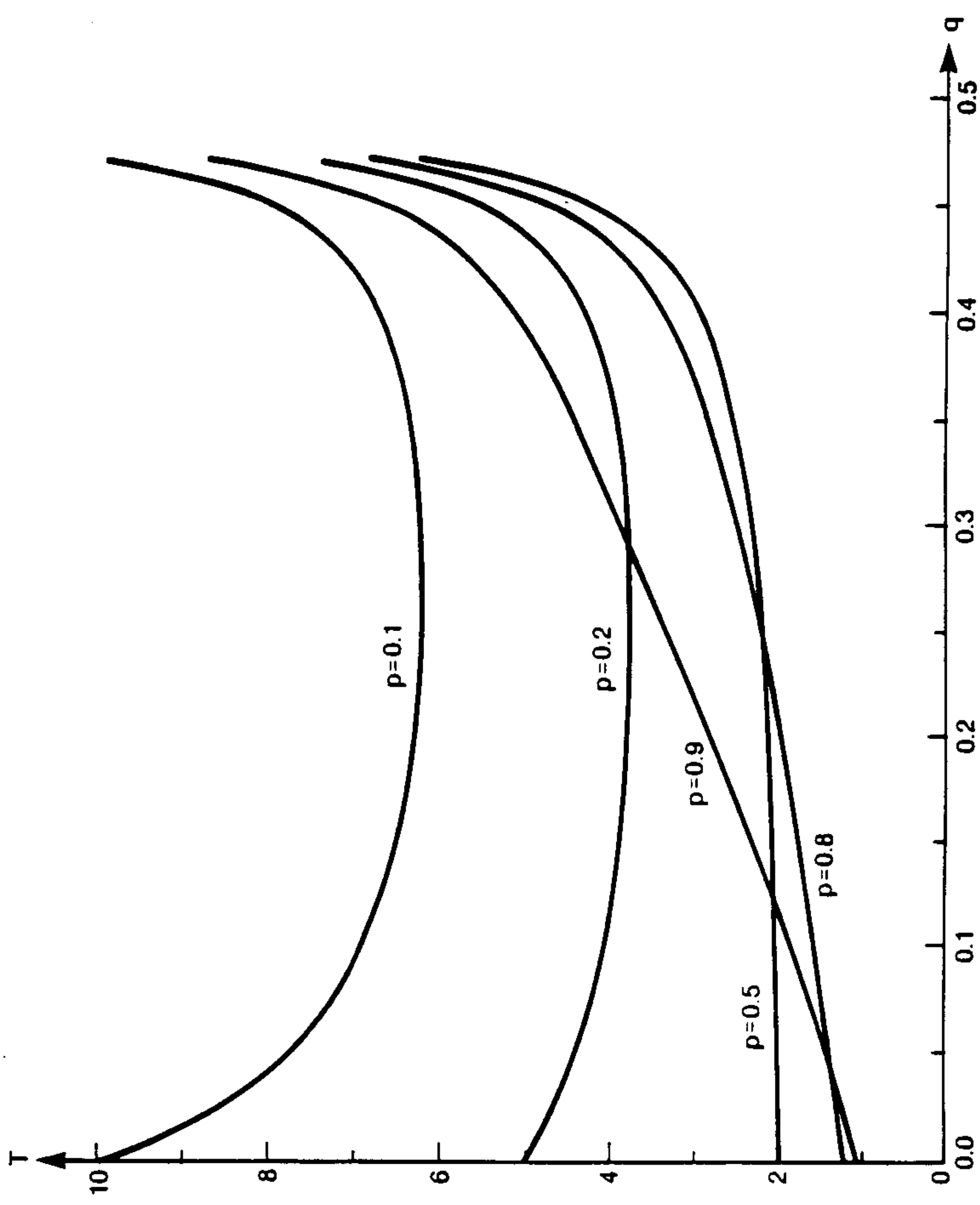

Figure 5.4: Polite system: The expected delay as function of the arrival rate

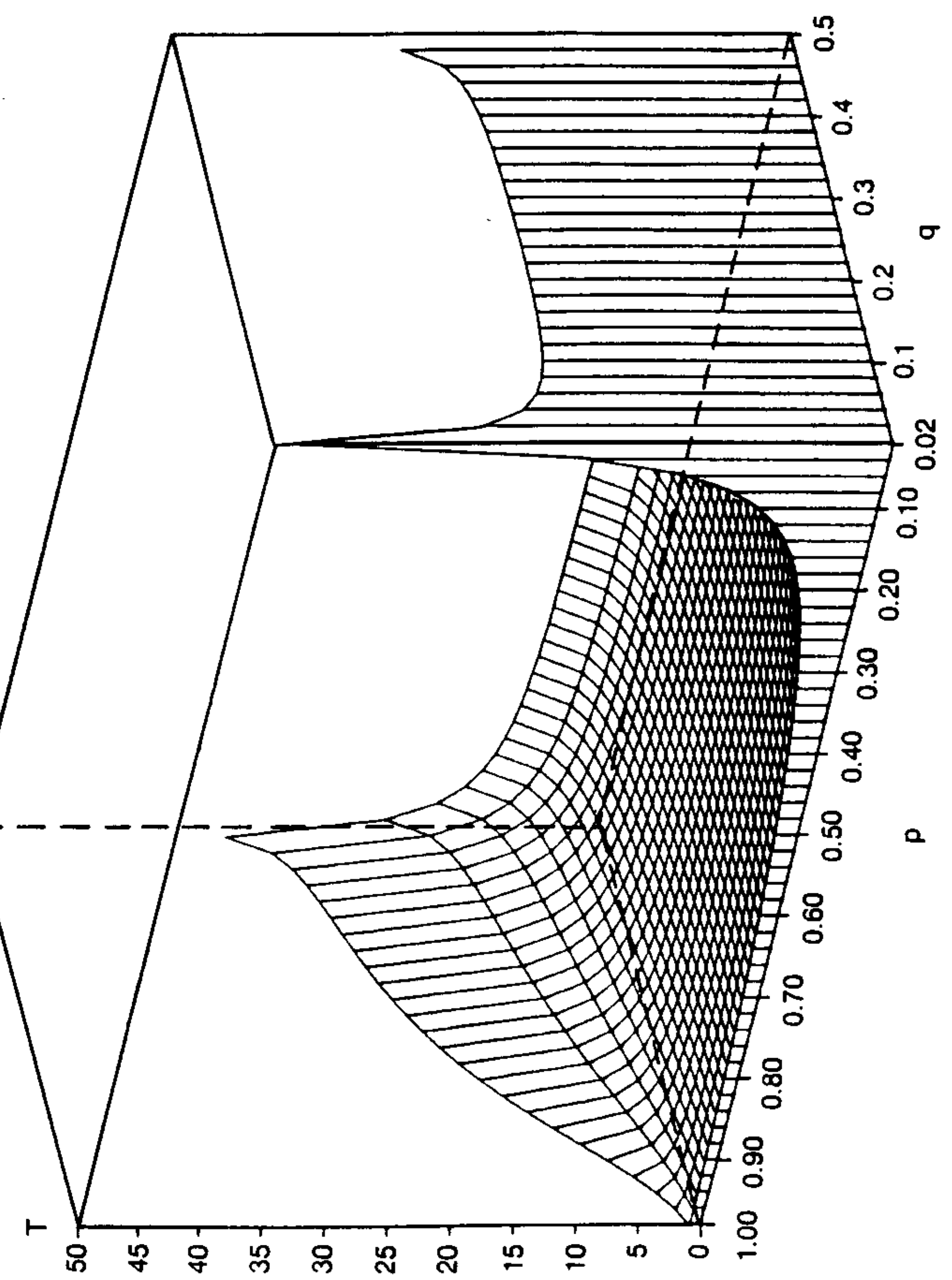

Figure 5.5: Polite system: The expected delay as function of the transmission probability and the arrival rat

5.5 Two Station Noisy Exhaustive Slotted ALOHA: The Queue with Starter Approach

In the following we analyze the expected delay in an exhaustive ALOHA system where the conflicts are resolved according to the *noisy* slotted ALOHA scheme. As in the analysis of the polite slotted ALOHA (section 5.4.1) we derive the expected delay of the two-station ALOHA system by analyzing the black box system and using the results from the analysis of the queue with starter. This analysis is done in two parts: First a fully symmetric system is analyzed, and then, the results are extended to a general (non symmetric) system. Note that for the noisy protocol the queue with starter analysis carries us all the way.

5.5.1 A Fully Symmetric System

We start this analysis by studying the fully symmetric ($q_i=q$, $p_i=p$; $i=1,2$) system. Due to the symmetry assumption, it is obvious that the only difference between this system and the system studied in section 5.4.1 above, is the behavior of the stations during the contention period. Thus, the original delay in the queue without starter is as reported in equation (5.7), and what is left to be done is to derive the additional delay suffered in the system due to the cold starts.

The behavior of an a noisy ALOHA system during the contention period can be one of two types:

a. Right before the beginning of the contention period exactly one station is willing to transmit (i.e., all the other stations are still idle). This station will transmit right after the end of the idle period, and this transmission will be successful. The length of the contention period, in this case is 0.

b. At the beginning of the contention period more then one station is willing to transmit. All these stations will transmit at the first slot of the contention period, and all these transmissions will collide. Following this collision slot, a station, say i, which was involved in the collision, will start transmitting at every slot with probability p_i. This will continue until one of the competing stations wins and successfully transmits a packet.

For our two-station system and for the corresponding black box system, we realize that possibility (a) occurs when the first slot following an idle period, contains exactly one customer. This corresponds to the case where the first arrival (following an idle period) to the two-station ALOHA system consists of one packet (which arrives either to station 1 or to station 2). The second possibility occurs when the first slot following an idle period in the black box system consists of two customers. This corresponds to the case where the first arrival to the two-station ALOHA system, consists of two packets: one arriving to station 1, and the other arriving to station 2.

From this observation it is easy to see that the length of the contention period is *not independent* of the arrival process. Thus, we can no longer apply the results of the queue with *independent* starter to our system. This is true, since in the analysis of the queue with independent starter (chapter 2) we assumed that the length of a cold start is *independent* of the arrival process following the cold start.

However, checking our system more carefully we observe that the length of the contention period depends *only* on the number of arrivals in the slot following the idle period. If the slot contains one customer, the length of the contention period is distributed as described in (a). If, on the other hand, this slot contains two packets, the length of the contention period is distributed according to the behavior described in (b). Therefore, this system corresponds to a queue with starter where the cold start depends on the number of customers arriving to the system at the beginning of a busy period. This type of a queue with starter was studied in chapter 3, and we can apply the results derived in that chapter to find the expected delay in our system.

Let us recall the main results reported in chapter 3 with regard to the queue with starter where the starter depends on the number of customers starting a busy period. According to these results, in order to calculate the expected value of the additional delay suffered by an arbitrary customer it is not sufficient to know the first two moments of the cold start distribution. In addition to these moments, one has also to calculate the *conditional moments* of the cold start distribution.

Let us recall the notation used in chapter 3:

$b_k \triangleq Pr[$ a busy period starts by the arrival of k customers $]$

$S(z \mid k) = S(z \mid$ the busy period starts by k customers $)$

$\overline{S_{|k}} \triangleq E[S \mid$ the busy period starts by k customers $]$

In chapter 3 it is shown that the expected value of the additional delay suffered by an arbitrary customer in the queue with starter can be calculated from $\overline{S}$, $\overline{S^2}$, $V(0)$, $\overline{S_{|k}}$ and b_k as following:

$$\overline{G} = \frac{1}{1+(1-V(0))\overline{S}} \cdot \left[\frac{(\overline{S^2}-\overline{S})(1-V(0))}{2} + \frac{\sum_{k=1}^{\infty} \overline{S_{|k}} \cdot b_k k}{\sum_{k=1}^{\infty} b_k k} \right] \tag{5.32}$$

Thus, in order to derive the additional delay suffered in our system we next calculate b_k, $S(z \mid k)$, $S(z)$, $\overline{S}$, $\overline{S^2}$, $\overline{S_{|k}}$, and $V(0)$:

1) b_k is the probability that a busy period in the queue without starter starts by the arrival of k customers. In our system a busy period may start either by one customer (the arrival of one packet to an empty system) or by two customers (the simultaneous arrival of two packets to an idle system). Thus, we can easily calculate:

$$b_1 = \frac{2q(1-q)}{2q(1-q)+q^2} = \frac{2-2q}{2-q}$$

$$b_2 = \frac{q^2}{2q(1-q)+q^2} = \frac{q}{2-q}$$

2) $S(z\,|k)$ is the z-transform of the cold start, conditioning on the number of customers starting the contention period. Thus we have:

$$S(z\,|1) = 1$$

$$S(z\,|2) = z\cdot\sum_{i=0}^{\infty}[2p(1-p)]\cdot[p^2+(1-p)^2]^i\cdot z^i = \frac{z\cdot 2p(1-p)}{1-[p^2+(1-p)^2]\cdot z}$$

3) $S(z)$ can be easily calculated from the above:

$$S(z) = b_1\cdot S(z\,|1)+b_2\cdot S(z\,|2) = \frac{2-2q}{2-q} + \frac{q}{(2-q)}\cdot\frac{z\cdot 2p(1-p)}{1-[p^2+(1-p)^2]\cdot z}$$

4) $\overline{S_{|1}}$ and $\overline{S_{|2}}$ can be calculated by differentiating $S(z\,|1)$ and $S(z\,|2)$ (respectively) with respect to z and evaluating the derivatives at $z=1$:

$$\overline{S_{|1}} = 0$$

$$\overline{S_{|2}} = \left.\frac{dS(z\,|2)}{dz}\right|_{z=1} = 1 + \frac{p^2+(1-p)^2}{2p(1-p)} = \frac{1}{2p(1-p)}$$

5) $\overline{S}$ can be calculated by differentiating $S(z)$ with respect to z and evaluating the derivative at $z=1$:

$$\overline{S} = \frac{q}{(2-q)}\cdot\frac{1}{2p(1-p)}$$

6) $\overline{S^2}-\overline{S}$ can be calculated by differentiating $S(z)$ twice with respect to z and evaluating the derivative at $z=1$:

$$\overline{S^2}-\overline{S} = \frac{d^2S(z)}{dz^2}\bigg|_{z=1} = \frac{q}{2-q}\cdot\frac{2p^2-2p+1}{2(p-1)^2p^2}$$

Next, we can substitute these expressions into equation (5.32) to give the expected value of the additional delay observed by an arbitrary customer in the system:

$$\overline{G} = \frac{q^2+2p(1-p)\cdot q(1-q)}{2p(1-p)[q^2+2p(1-p)]} \tag{5.33}$$

Now we can add the original delay observed in the system (calculated in equation (5.7)) to the additional delay derived above and get the expected value of the total delay suffered in the system:

$$T = 1 + \frac{q}{2(1-q)} + \frac{q^2+2p(1-p)\cdot q(1-q)}{2p(1-p)[q^2+2p(1-p)]} \tag{5.34}$$

This is the system time in the two-station exhaustive noisy ALOHA system.

Optimization of the Transmission Probability

An important issue in planning access schemes is to optimally set the values of the system parameters. In this system, it is important to set p (the symmetric transmission probability, used by both stations) to minimize the expected delay. It is obvious, that in order to find the optimal value of p, one does not have to calculate the value of the total delay. Since p affects only the expected value of the additional delay,and since the total delay is the sum of the original delay and the additional delay, then one can find the optimal value of p by analyzing the expression of the additional delay.

For the sake of optimization we can substitute $2p(1-p) = x$ into equation (5.33), giving:

$$\overline{G} = \frac{q^2+xq(1-q)}{x(q^2+x)}$$

It can be seen that this expression monotonically decreases for $x \geq 0$, so the minimal value of $\overline{G}$ is achieved for the maximal value of x ($= 2p(1-p)$). This maximum, under the constraint that $0 \leq p \leq 1$ is achieved when $p=1/2$. Thus, the optimal symmetric transmission probability is $p=1/2$. This result is quite intuitive since $p=1/2$ is the symmetric transmission probability which will minimize the length of the contention period.

Now, substituting $p=1/2$ into (5.33) we get the expected value of the additional delay under the optimal policy:

$$\overline{G}_{opt} = \frac{2q^2+2q}{2q^2+1} \tag{5.35}$$

Accordingly, the optimal system time is:

$$T_{opt} = 1 + \frac{q}{2q(1-q)} + \frac{2q^2+2q}{2q^2+1} \tag{5.36}$$

5.5.2 A Non Symmetric System

Having analyzed the fully symmetric system we next study the non symmetric system. We assume that the arrival rate to stations 1 is q_1 and the arrival rate to station 2 is q_2 (both arrival streams are Bernoulli independent as in the previous models). The transmission probability used by station 1 during the conflict resolution period is p_1, and the transmission probability used by station 2 is p_2.

The approach to analyze this system is identical to the one used in sub-section 5.5.1 above, and the notation to be used in the sequel is identical to the notation used in sub-section 5.5.1.

Following the calculation in section 5.5.1 we first calculate the additional delay in the system. This is calculated as follows:

1) The value of b_k is:

$$b_1 = \frac{q_1 + q_2 - 2q_1q_2}{q_1 + q_2 - q_1q_2}$$

$$b_2 = \frac{q_1q_2}{q_1 + q_2 - q_1q_2}$$

2) The value of $S(z\ |k)$ is:

$$S(z\ |1) = 1$$

$$S(z\ |2) = \frac{z[p_1(1-p_2)+p_2(1-p_1)]}{1-[1-p_1-p_2+2p_1p_2]\cdot z}$$

3) The value of $S(z)$ is:

$$S(z) = = \frac{q_1 + q_2 - 2q_1q_2}{q_1 + q_2 - q_1q_2} + \frac{q_1q_2}{q_1 + q_2 - q_1q_2} \cdot \left(\frac{z[p_1(1-p_2)+p_2(1-p_1)]}{1-[1-p_1-p_2+2p_1p_2]\cdot z} \right)$$

4) The value of $\overline{S_{|k}}$ is:

$$\overline{S_{|1}} = 0$$

$$\overline{S_{|2}} = \frac{1}{p_1+p_2-2p_1p_2}$$

5) The value of $\overline{S}$ is:

$$\overline{S} = \frac{q_1q_2}{q_1+q_2-q_1q_2} \cdot \frac{1}{p_1+p_2-2p_1p_2}$$

6) The value of $\overline{S^2}-\overline{S}$ is:

$$\overline{S^2}-\overline{S} = \frac{q_1q_2}{q_1+q_2-q_1q_2} \cdot \frac{2(1+2p_1p_2-p_1-p_2)}{(p_1+p_2-2p_1p_2)^2}$$

In addition let us calculate $V(0)$, the probability that no packet arrives to the system at time t. This is:

$$V(0) = (1-q_1)(1-q_2)$$

Now substituting the values calculated above into equation (5.32) yields the expected value of the additional delay suffered in the system:

$$\overline{G} = \frac{q_1q_2[2(p_1+p_2-2p_1p_2)+(q_1+q_2)(1+2p_1p_2-p_1-p_2)]}{(p_1+p_2-2p_1p_2+q_1q_2)(p_1+p_2-2p_1p_2)(q_1+q_2)} \qquad (5.37)$$

In addition to the calculation of the additional delay we have to recalculate the original delay, suffered in the queue without starter. To derive this value we repeat the derivation of equation (5.7) but with non symmetric arrivals. This calculation yields:

$$E[\text{ system time in the queue without starter }] = 1 + \frac{q_1q_2}{(1-q_1-q_2)(q_1+q_2)} \qquad (5.38)$$

Adding the original delay to the additional delay, finally gives the expected delay in the non symmetric system:

$$T = 1 + \frac{q_1q_2}{(1-q_1-q_2)(q_1+q_2)} + \frac{q_1q_2[2(p_1+p_2-2p_1p_2)+(q_1+q_2)(1+2p_1p_2-p_1-p_2)]}{(p_1+p_2-2p_1p_2+q_1q_2)(p_1+p_2-2p_1p_2)(q_1+q_2)} \qquad (5.39)$$

It is obvious that the set of values for p_1, p_2 which minimizes the expected delay is the set that maximizes the probability of successful transmission when both stations are willing to transmit. This set (see[Yemi80]) is either $\{p_1=1,\ p_2=0\}$ or $\{p_1=0,\ p_2=1\}$.

5.6 Discussion of the Results for Symmetric Two Station Systems

In this section we compare the expected delay as observed in three systems: 1) The non exhaustive slotted ALOHA scheme, described in section 5.1. 2) The noisy exhaustive ALOHA scheme. 3) The polite exhaustive ALOHA scheme. The comparison is done for the results derived for the symmetric system.

5.6.1 A Comparison of the Results

As we studied in the previous sections, the expected delay in each of these schemes depends both on the transmission probability p and on the arrival rate (to a single queue) q. In the following we compare these schemes when they are *optimally operated*. A scheme is optimally operated if for every arrival rate q the transmission probability p is chosen to minimize the expected delay. An *optimal transmission policy* for a given scheme, is a function $F(q)$, which defines for every arrival rate q the optimal transmission probability to be used: $p_{opt} = F(q)$.

The first question of interest is to compare the optimal transmission policies of these schemes. Equation (5.2) of section 5.1 gives the analytic expression for the optimal transmission policy for the non exhaustive scheme. From section 5.5.1 it is obvious that the optimal transmission policy for the noisy exhaustive ALOHA is: $p_{opt} = 1/2$, independent of the arrival rate. Thus, it only remains to derive the optimal transmission policy of the polite exhaustive ALOHA. Due to the complexity of the expression for the expected delay in this system, we are unable to derive an analytic expression for the optimal transmission policy of this scheme; we use numerical methods instead.

In figure 5.6 we plot the optimal transmission policies of the three schemes as a function of the arrival rate q. The optimal policies for the exhaustive schemes are plotted over the range $0 \leq q \leq 1/2$. The optimal policy for the non-exhaustive scheme is plotted only over the range $0 \leq q \leq 1/4$, which is the stable range for this scheme.

From this figure we can observe the following properties:

i. The optimal transmission probability, in any of the schemes is non increasing function of q. This is true since in any scheme, higher loads make the queues less likely to be empty, which in turn, suggests a more polite (lower p) policy.

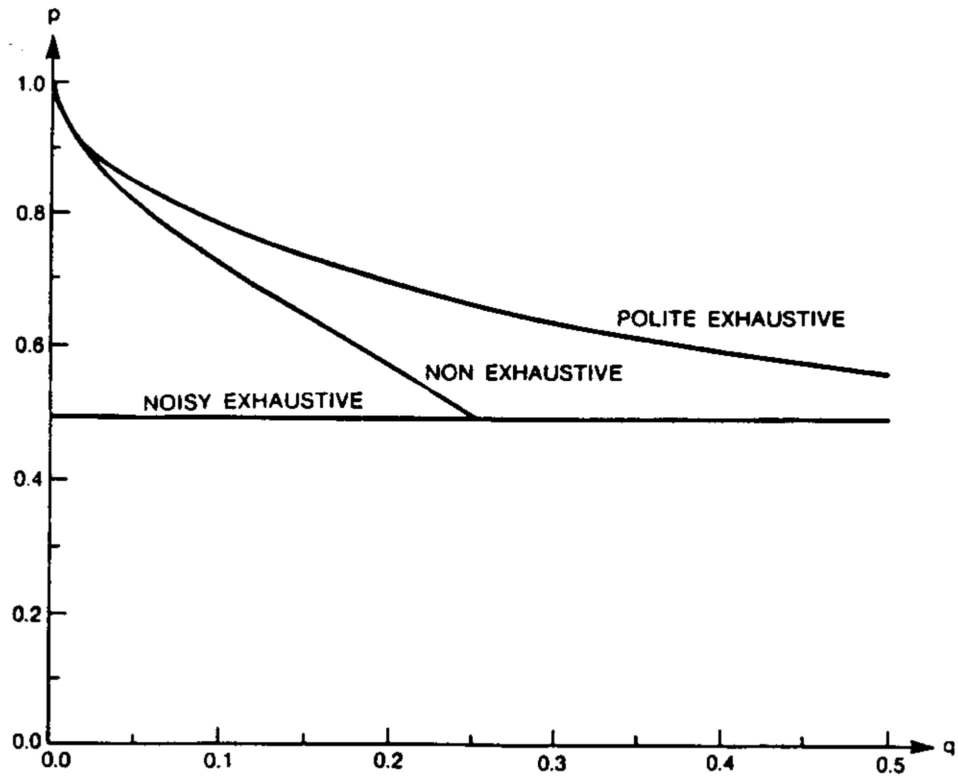

Figure 5.6: Two stations: The optimal transmission probability as function of the arrival rate

ii. The optimal transmission probability of the polite exhaustive scheme is always higher than the optimal transmission probability of the non exhaustive scheme. The reason for this property, is that at any arrival rate a queue in the non exhaustive scheme is less likely to be empty than a queue in the exhaustive polite scheme.

iii. Only for one value of q ($q = 1/4$) does the optimal transmission probability of the non exhaustive scheme equal to one half. The reason is that only at this load is the system heavily loaded, and none of the queues gets empty. Thus, in this case the heavy load approximation holds, and the optimal transmission policy is 1/2. For any other value of q, the system is not heavily loaded, and the optimal transmission probability must be higher.

iv. The optimal transmission probability of the polite exhaustive scheme is always greater than one half. The reason is that a contention period in this system always follows an idle period. Thus, the stations *can not* be assumed to be heavily loaded during the contention period, and the optimal transmission probability must be greater than a half.

Next we compare the expected delay observed in these systems. Figure 5.7 depicts the expected delay in the three systems when they are operated under the optimal transmission policy. In addition, for comparison, we plot in this figure the expected delay in a single server queue (no contention) which operates under the same arrival process (the expected delay as given in equation (5.7)).

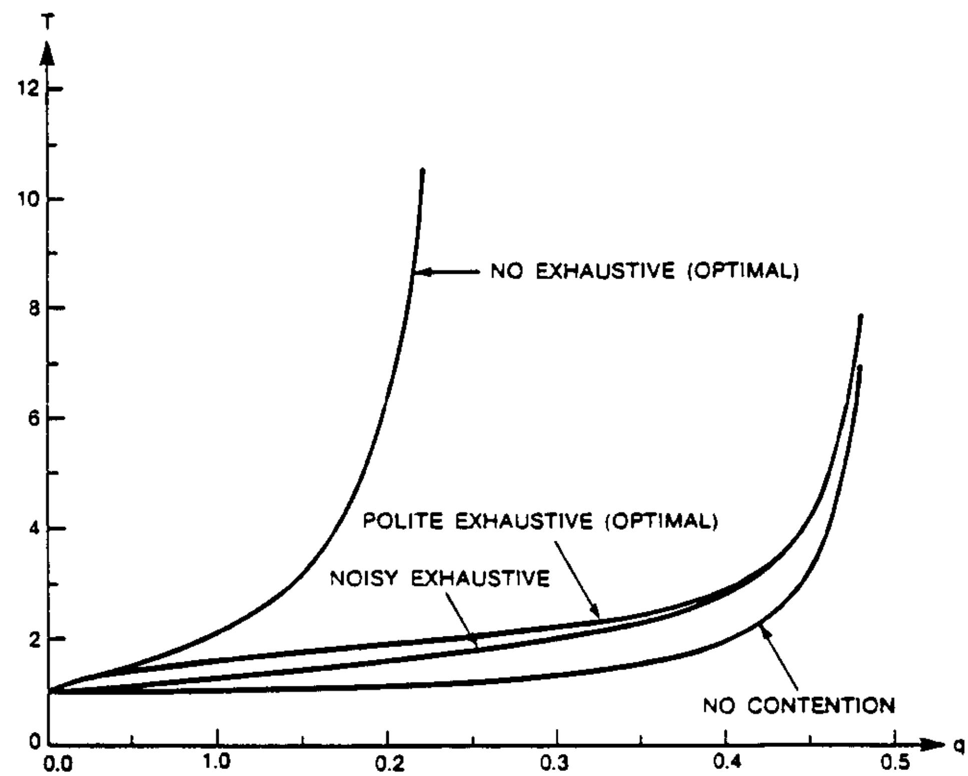

Figure 5.7: The optimal (minimal) expected delay as function of the arrival rate

It is easy to see that for every arrival rate q, both exhaustive schemes are superior to the non exhaustive scheme. Similarly, for every arrival rate q, the single server queue (without contention) is superior to both exhaustive schemes. Both these observations are very intuitive.

A more surprising observation is that the expected delay of the exhaustive schemes is almost identical to the expected delay of the single server queue. To test this observation more closely, we plot in figure 5.8 the difference between the expected delay in the exhaustive scheme to the expected delay in the non-contention single server system. This can be thought of as an extra delay (i.e. the *additional delay)* spent in the ALOHA schemes due to the *distributed control*

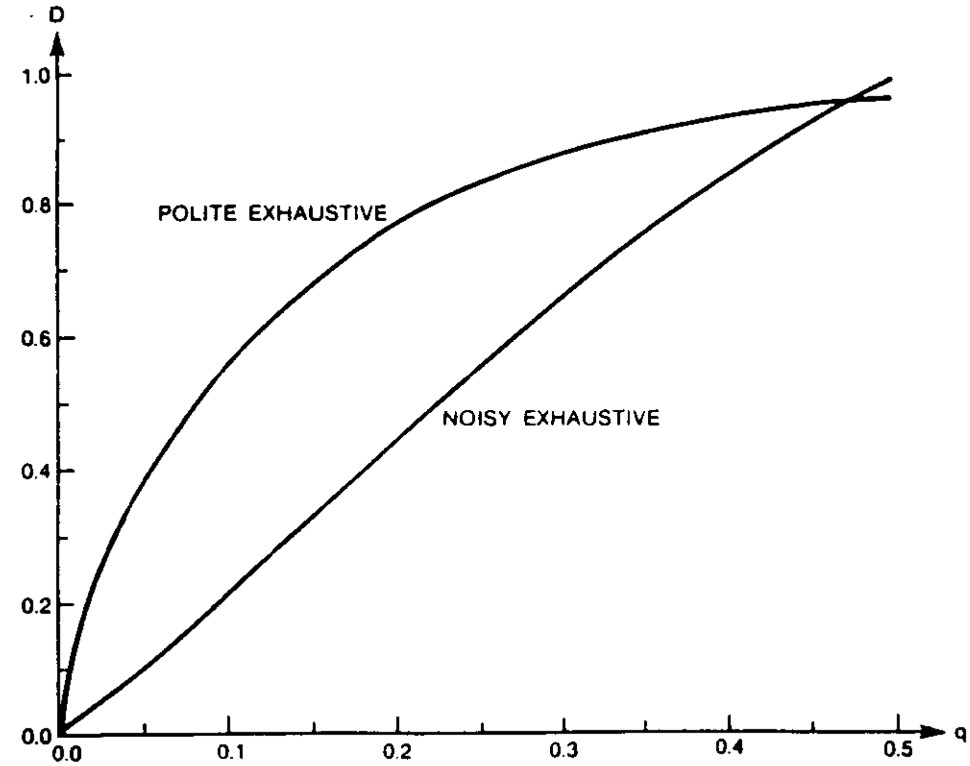

Figure 5.8: The additional delay observed in the exhaustive schemes

they use. From figure 5.8 we see that the performance of the exhaustive schemes (in terms of expected delay) is very close to the performance of an optimally controlled scheme (namely, the single server queue) and the "extra delay" suffered in these schemes is bounded by a unity. The fact that the optimal additional delay suffered in the polite scheme is bounded by 1 agrees with our previous results; according to these results for $p=1/2$ the additional delay is exactly 1. Thus, the additional delay in an optimally controlled scheme cannot be greater than 1.

5.6.2 Applications

In the analysis carried in the previous sections the measure for the expected delay was the time slot. Thus, in the comparison made above, we compared the expected number of slots a packet stays in the system under different transmission policies.

Recalling the description of the exhaustive ALOHA protocol, we realize that an extra bit of every packet is used as an end-of-use flag. Such a flag in the non exhaustive scheme is not required. For this reason is it clear that the time duration of a slot in the exhaustive scheme is slightly longer than the time duration of a slot in the non exhaustive scheme.

Let b be the number of bits used in a packet in the non exhaustive scheme. Thus, the number of bits that must be used in the exhaustive schemes is $b+1$, and in order to compare the delay of the different schemes one must multiply the delay suffered in the exhaustive scheme by a factor of $1+1/b$.

It is obvious that for any reasonable packet size (e.g., 100 bits and above) this factor is very small, and does not affect most of the results. The two effects this factor has on the results discussed above are:

1. At very low arrival rates ($q \rightarrow 0$) the delay in the non exhaustive scheme will be lower than the delay in the exhaustive schemes.

2. The additional delay suffered in the exhaustive schemes (in comparison to the expected delay suffered in the single server queue) is not bounded by a constant*.

Nevertheless, these two effects are minor and for most practical purposes the results reported above are accurate enough.

Another important issue is the applicability of the exhaustive scheme as described in this section. In all the analysis done above we used an important property of the two-station exhaustive scheme: When one station finishes transmitting, the other station can start transmitting in the next slot without any interruption. This assumption cannot be applied to an N-station system, since in such a system, when one station finishes transmitting, several other stations may be willing to transmit.

For this reason we conclude that the analysis done above holds *only* for two-station systems, where the stations *know* that they are the only stations in the system. Therefore, if one is willing to analyze a two-station exhaustive system, where the two stations *cannot assume* that they are the only stations in the system, one cannot use this method. However, for such systems, the expected delays derived above can be used as a lower bound approximation of the actual expected delay in the system. This we investigate in the following section.

*It is obvious that under heavy load the cost of the end-of-use flag is very high. Clearly, at very high loads it is more efficient to use the "silent slot" method to notify the end of transmission.

5.7 N Symmetric Stations: An Approach by the Random Polling System

In this section we study the expected delay in an exhaustive ALOHA system which consists of N queues. It is obvious that the events in such a system are strongly correlated to each other, so there is little hope of a closed form exact delay analysis of the system.

Our goal is to approximate the expected delay in this system under certain conditions. This analysis is carried out under the basic assumption that the stations are symmetric. Thus, it is assumed that the arrival process to each station is a Bernoulli process with parameter q, and that the conflict resolution scheme, used during the contention periods, is the same for all stations.

The main analysis tool, used in this section, is the *exhaustive service random polling system,* studied in chapter 4. The reason for this is that the behavior of the exhaustive ALOHA system is very similar to that of the random polling scheme: 1) The service method in both systems is exhaustive. 2) The contention period in the ALOHA scheme can be approximately modeled as the switch over period in the random polling system. 3) In the random polling system the station to be served following a switch over period is chosen among all the stations by a random method. In the exhaustive ALOHA scheme the station to be served following a contention period, is randomly selected among all the stations who participated in the contention period. Therefore, our analysis method in this section is to emulate the ALOHA scheme by the proper random polling system, and to use the results derived in chapter 4.

Before starting the analysis, some more definitions are required. In section 5.2 we have classified the slots in the exhaustive ALOHA scheme into three types: 1) Transmission slots. 2) Contention slots. 3) Idle slots. In addition to the above, a slot is called *a potential transmission slot* if it is either a contention slot or the first slot in the transmission period. The common property of all the potential transmission slots, is that at the beginning of such a slot there are several stations (at least one) who are trying to transmit in the slot. If exactly one of these stations transmits at this slot, then the slot is a transmission slot; otherwise, it is a contention slot.

In addition, let us recall from chapter 4 some of the notation used in the analysis of the random polling systems. The notation used to denote the arrival process is:

$$X_i(t) \triangleq \text{the number of customers arriving to station } i \text{ at time } t$$

$$\mu \triangleq E[X_i(t)] \quad ; \quad \sigma^2 \triangleq Var[X_i(t)]$$

The length of a switch over period is denoted by R and we have:

$$r \triangleq E[R] \quad ; \quad \delta^2 \triangleq Var[R]$$

5.7.1 A Heavy Load Approximation of the Polite Exhaustive ALOHA

In the following we approximate the expected delay in a polite exhaustive ALOHA system, by the expression of the expected delay in the exhaustive service random polling system. In order to use this approximation, we have to define the behavior of the random polling system, in a way that will simulate the ALOHA system. Since in both systems the service policy is exhaustive service, what is left to be done is to simulate the contention period of the ALOHA system by the switch over period of the random polling system. Thus we will extract parameters from the exhaustive slotted ALOHA system and substitute them into the delay expressions of the random polling system.

First, we examine the behavior of the ALOHA system during the contention period. Since this system is assumed to be heavily loaded, it can be assumed that at the end of an exhaustive service period exactly $N-1$ of the stations are busy, and one of them, say station i, (the station which was just served) is idle. Clearly, this claim is not correct in all cases (there are still occasions where the whole system is idle!) but it can be used as a good approximation of the system status in most cases. Therefore, the number of busy stations, at each contention slot, say t, is either $N-1$ or N. In the first case, no packet arrives to station i during the contention period before time t, and in the second case at least one packet arrives to station i during this period.

It is clear that the time at which the first packet arrives to station i affects both the length of the contention period and the performance of the system. For simplicity of the analysis, and since we are interested only in an approximation, let us consider two simple extreme cases:

1. The first packet arriving to station i arrives right before the end of the exhaustive service period. Thus, the number of busy stations during the whole duration of the contention period is N. It is obvious that this is not a feasible event in the exhaustive scheme, since the first packet can arrive to station i only after the contention period started However, this can serve as a simple extreme case for our approximation.

2. No packet arrives to station i, during the duration of the contention period. Thus the number of busy stations, during the contention period is always $N-1$.

We start the analysis by looking at the first case. Let s_A denote the probability that a potential transmission in the slotted ALOHA system is a successful transmission, i.e.:

$$s_A \triangleq Pr[\text{ slot } t \text{ is a successful transmission } \mid \text{ slot } t \text{ is a potential transmission }]$$

Under the assumption that at any contention slot all the N stations are willing to transmit, it is clear that for every potential transmission slot we have:

$$s_A = Np(1-p)^{(N-1)} \tag{5.40}$$

Let r_i denote the probability that the length of a switch over period, in the random polling system is i:

$$r_i \triangleq Pr[R=i]$$

Using the above notation, we can simulate the exhaustive ALOHA system by an exhaustive random polling system where we have:

$$r_i = (1-s_A)^i \cdot s_A \quad ; \qquad i=0,1,2,\cdots \tag{5.41}$$

From (5.41) it is easy to derive the expected value and the variance of the length of the switch over period:

$$r = E[R] = \frac{(1-s_A)}{s_A} \quad ; \qquad \delta^2 = Var[R] = \frac{(1-s_A)}{{s_A}^2} \tag{5.42}$$

Now we must find the parameters of the arrival process; since the arrival process to each station is a Bernoulli process with parameter q we have:

$$\mu = E[X_i(t)] = q \quad ; \qquad \sigma^2 = Var[X_i(t)] = q(1-q) \tag{5.43}$$

Now, to derive the expected waiting time in the system, we substitute μ, σ^2, r and δ^2 into equation (4.54?), to yield:

$$T = \frac{1}{2s_A} + \frac{1-q}{2(1-Nq)} + \frac{N[\frac{1}{s_A}-1]\cdot(1-q)}{2(1-Nq)} + \frac{(N-1)\cdot[\frac{1}{s_A}-1]}{2(1-Nq)} \tag{5.44}$$

where:

$$s_A = Np(1-p)^{(N-1)}$$

For the second case we calculate the delay in a system where exactly $N-1$ of the stations take part in the contention period. The simulation of this system by the random polling system is not as straight forward. The reason is that in the random polling system the next station to be served after a switch over period is selected among all the stations, while in the exhaustive ALOHA system the next station to be served is selected from the $N-1$ busy stations.

Let us define s_A as defined above, namely, this is the probability that a potential transmission slot in the slotted ALOHA system is a successful transmission slot. It is clear that in this system the value of s_A, for every potential transmission slot, is:

$$s_A = (N-1)p(1-p)^{(N-2)} \tag{5.45}$$

And the length of a contention period is geometrically distributed with parameter s_A.

Let R be a random variable representing the length of a switch over period in the random polling system, and let $R(z)$, r and δ^2 be the z-transform, the mean and the variance of R, respectively.

Let us denote by the *virtually idle period* (of the polling system), the period that starts at the beginning of any switch over period, and ends right before the beginning of the service period (i.e., right before a customer is first served following the switch over period). We call this period the virtually idle period since during this time the server is idle, but the system is not necessarily idle. Let R_1 be the random variable representing the length of the virtually idle period, and let $R_1(z)$, r_1 and δ_1^2 be the z-transform, the mean and the variance of R, respectively.

It is obvious that in the simulation the contention period of the slotted ALOHA system should be represented by the virtually idle period of the random polling system (and not simply by the switch over period!), since this is the period during which the server is idle in these systems. Thus, R_1 in the simulation is distributed as the length of the contention period, and in order to complete the simulation it is now required to calculate r and δ^2 (from r_1 and δ_1^2) and substitute them in the expression of the expected delay in the random polling system.

Under the assumption that exactly N-1 stations are busy during the switch over period, it is easy to derive a relation between $R_1(z)$ and $R(z)$. This is:

$$R_1(z) = \frac{N-1}{N}\cdot R(z) + \frac{1}{N}R(z)\cdot R_1(z) \tag{5.46}$$

The explanation of this expression is simple: The first term in the expression represents the case where at the end of a switch over period the server selects one of the N-1 busy stations. In this case the length of the virtually idle period is identical to the length of the switch over period. The second term in the expression represents the case where at the end of the contention period the server selects the idle station. In this case the length of the virtually idle period is distributed as the sum of two independent random variables: 1) the length of a switch over period. 2) the length of a virtually idle period.

From (5.46) we derive:

$$R(z) = \frac{NR_1(z)}{N-1+R_1(z)} \tag{5.47}$$

Now, we calculate r and δ^2:

$$r = \frac{dR(z)}{dz}\bigg|_{z=1} = \frac{(N-1)r_1}{N} \tag{5.48}$$

$$\frac{d^2R(z)}{dz^2}\bigg|_{z=1} = \frac{N-1}{N}\cdot(\delta_1^2+r_1^2-r_1) - \frac{2(N-1)r_1^2}{N^2}$$

so we have:

$$\delta^2 = \frac{N-1}{N}\cdot\left[\delta_1^2 - \frac{r_1^2}{N}\right] \tag{5.49}$$

As stated above, the length of the virtually idle period is distributed as the length of the contention period. Thus, we have:

$$Pr[R_1 = i] = (1-s_A)^i \cdot s_A \quad ; \quad i = 0,1,2,\cdots$$

where:

$$s_A = (N-1)p(1-p)^{N-2}$$

Thus:

$$r_1 = \frac{1-s_A}{s_A} \quad ; \quad \delta_1^2 = \frac{1-s_A}{s_A^2} \tag{5.50}$$

so we have:

$$r = \frac{N-1}{N}\cdot\frac{1-s_A}{s_A} \tag{5.51}$$

$$\delta^2 = \frac{N-1}{N}\cdot\frac{1-s_A}{s_A^2}\cdot\left(\frac{(N-1)+s_A}{N}\right) = r(1+r) \tag{5.52}$$

Finally, we substitute μ, σ^2, r and δ^2 into equation (4.54?) to derive the expected delay in the ALOHA system:

$$T = \frac{1+r}{2} + \frac{1-q}{2(1-Nq)} + \frac{Nr(1-q)}{2(1-Nq)} + \frac{(N-1)r}{2(1-Nq)} \tag{5.53}$$

where:

$$r = \frac{1}{Np(1-p)^{N-2}} - \frac{N-1}{N}$$

Discussion

Two different expressions have been suggested above to approximate the expected delay in the N-station polite exhaustive ALOHA system for heavy traffic. The first expression assumes that all the N stations compete during the contention period and the second expression assumes that exactly $N-1$ of the stations compete during this period. The first expression tends to be a pessimistic approximation while the second expression tends to be an optimistic approximation. The true value of the expected delay is likely to lie somewhere between the two values given by these expressions.

When the number of stations is small (small N) it is reasonable to believe that the pessimistic expression is a better approximation than the optimistic approximation. The reason is that the arrival rate to a single queue, when the system is at heavy load, is about $\frac{1}{N}$. When N is small, $\frac{1}{N}$ is a relatively high arrival rate so it is very likely that the station which was idle at the beginning of the contention period will soon get busy (due to a new arrival to this station) and the contention in most slots will be among N stations. On the other hand, when N gets larger, the arrival rate to a single station, when the system is at heavy load, is very low, and the idle station is very unlikely to become busy during the contention period. Thus, when the number of stations is large, the pessimistic expression (assuming that exactly $N-1$ stations are willing to transmit during the contention period) is believed to be a good approximation of expected delay in the system. Nevertheless, note that for large values of N the two delay expressions are approximately the same, so both can be used to approximate the expected delay in the system.

An Invariant Behavior under Heavy Load

As described above the length of a contention period when the system is at heavy load, may depend on the time at which the first packet arrives to the idle station. The probability that a potential transmission slot is a successful transmission when N stations participate in the contention period is:

$$Np(1-p)^{N-1}$$

The probability that a potential transmission slot is a successful transmission slot when $N-1$ stations participate in the contention period is:

$$(N-1)p\cdot(1-p)^{N-2}$$

Let us examine what is the transmission probability p for which the behavior of the polite exhaustive ALOHA during the contention period (at heavy load) is independent of the number of stations (N or $N-1$) which participate in the contention period. Let p^* be this invariant transmission probability. Thus, we must have:

$$Np^*(1-p^*)^{N-1} = (N-1)p^*\cdot(1-p^*)^{N-2}$$

or:

$$p^* = \frac{1}{N}$$

This is exactly the transmission probability which maximizes the system throughput when the system is under heavy load (see section 5.1 above).

It is clear that under these conditions the length of the contention period is independent of the time at which a new packet arrives to the idle station. Thus, if the transmission probability is chosen to maximize the heavy load throughput, then the length of the contention period in the polite exhaustive ALOHA system (when the system is at heavy load) is geometrically distributed with parameter:

$$Np^*(1-p^*)^{N-1} = (1-\frac{1}{N})^{N-1}$$

5.7.2 A Heavy Load Approximation of the Noisy Exhaustive ALOHA

The approximation of the expected delay in the noisy exhaustive ALOHA system is similar to the approximation of the polite system. Under heavy load conditions, the number of stations which are busy right after the end of a contention period is exactly $N-1$. Thus, the first slot of the contention period is a collision slot (of $N-1$ stations)* and the rest of the period consists of contention slots. In each of these contention slots exactly $N-1$ stations are willing to transmit. Thus, the probability of successful transmission in each of these slots is:

$$s_A = (N-1)p(1-p)^{N-2} \tag{5.54}$$

If we assume that the idle station becomes busy before the end of the contention period, then as in the polite system, we can simulate the ALOHA system by a random polling system where the length of the switch over period is distributed as the length of the contention period. Thus, the switch over period is distributed as:

$$r_i = s_A \cdot (1-s_A)^{i-1} \quad ; \qquad i=1,2,\cdots \tag{5.55}$$

The expected value and the variance of this distribution are:

$$r = \frac{1}{s_A} \quad , \qquad \delta^2 = \frac{(1-s_A)}{{s_A}^2} \tag{5.56}$$

and the expected delay, in the system is:

$$T = \frac{1-s_A}{2s_A} + \frac{1-q}{2(1-Nq)} + \frac{N[\frac{1}{s_A}]\cdot(1-q)}{2(1-Nq)} + \frac{(N-1)\cdot[\frac{1}{s_A}]}{2(1-Nq)} \tag{5.57}$$

where the value of s_A is given in equation (5.54).

*This observation is true only when $N>2$. For the analysis of a two-station system, see section 5.8.

If, on the other hand, we use the optimistic assumption, namely, if we assume that the idle station does not become busy before the end of the contention period, then we follow the derivation for the similar case in the polite system. Thus we get that the length of the virtually idle period in the polling system is distributed as:

$$Pr[R_1 = i] = s_A \cdot (1-s_A)^{i-1} \quad ; \qquad i=1,2,\cdots \tag{5.58}$$

and the mean and the variance of the virtually idle period are:

$$r_1 = \frac{1}{s_A} \quad ; \qquad \delta_1^2 = \frac{1-s_A}{s_A^2} \tag{5.59}$$

where s_A is given in equation (5.54).

From equations (5.59), (5.48) and (5.49) we can derive the mean and the variance of the corresponding switch over period:

$$r = \frac{N-1}{Ns_A} \quad ; \qquad \delta^2 = \frac{N-1}{Ns_A} \cdot \left[\frac{(N-1)-Ns_A}{Ns_A} \right] = r(r-1) \tag{5.60}$$

and the expected delay in the system is:

$$T = \frac{r-1}{2} + \frac{1-q}{2(1-Nq)} + \frac{Nr(1-q)}{2(1-Nq)} + \frac{(N-1)r}{2(1-Nq)} \tag{5.61}$$

where the value of r is:

$$r = \frac{1}{Np(1-p)^{N-2}}$$

As in the polite exhaustive ALOHA system, the pessimistic approximation is good for systems with a small number of stations, and the optimistic approximation is good for systems with a large number of stations.

5.8 Two Stations in a General N station Environment

The two-station system studied in sections 5.3, 5.4, 5.5 and 5.6 above, were studied under the basic assumptions that the two stations *know* and *can assume* that they are the only stations willing to transmit in the system. This assumption was used in the access scheme employed by the station, and allowed the continuous transmission of packets in the system as long as any of the stations has something to transmit.

In the following we study the behavior of a two-station system where the stations cannot assume that they are the only stations willing to transmit in the system. In this system, after a transmission period of station 1, station 2 cannot assume that it is the only station in the system who wants to transmit (if station 2 has a packet to transmit at this moment), so it cannot transmit right after hearing the end-of-use flag transmitted by station 1. Rather, a collision resolution scheme has to be applied before the beginning of every transmission period.

5.8.1 Polite Exhaustive ALOHA: An Approximation

It is obvious that the expected delay in the two-station system, studied in sections 5.3, 5.4 5.5 and 5.6 above, is a lower bound for the expected delay in this system. Thus, one can use the expressions (5.9) and (5.26) as a lower bound for the delay in this system. In addition, it can be noted that the behavior of the studied in sections 5.3, 5.4 5.5 and 5.6 is similar to the behavior of the system we study here when the arrival rate is low. In this case, transmission periods are relatively short, and it is very rare that station 2 will be busy when station 1 finishes transmitting. Thus, this lower bound is a good approximation of the expected delay when the system load is low. On the other hand, when the arrival rate is high, this lower bound cannot be used as a good approximation of the system delay. The reason is that at high load, station 2 is very likely to be busy when station 1 finishes transmitting, and the similarity between the systems is very weak.

Thus, we conclude that the expected delay derived in sections 5.4 and 5.5 is a lower bound for the expected delay in our system, and that these results are good approximations of the expected delay in our system when the system is under light load.

At the other extreme we have the analysis of section 5.7 (using the random polling method). This analysis provides an heavy load approximation of the expected delay in the system. As stated in section 5.7, when the number of stations is small (and this is the case here) a good approximation of the heavy load delay is taken from equation (5.44). This expression, when evaluated at $N=2$ yields:

$$T = \frac{1}{2s_A} + \frac{1-q}{2(1-Nq)} + \frac{2[\frac{1}{s_A}-1]\cdot(1-q)}{2(1-Nq)} + \frac{[\frac{1}{s_A}-1]}{2(1-Nq)} \qquad (5.62)$$

where:

$$s_A = 2p(1-p)$$

Using the two approximations mentioned above, let us now approximate the expected delay in the system when the transmission probability p, is the invariant transmission probability; i.e., $p=1/2$. The light load approximation is given by equation (5.11):

$$T_{\text{light}} = 2 + \frac{q}{2-4q} \quad (5.63)$$

The heavy load approximation is calculated by substituting $p=1/2$ in equation (5.62):

$$T_{\text{heavy}} = 2 + \frac{q}{2-4q} + \frac{2}{2-4q} \quad (5.64)$$

Now, if one is interested in approximating the expected delay in the system over the whole range of arrival rates ($0 \leq q \leq 1/2$), a natural approach is to use a linear combination of equations (5.63) and (5.64). This combination yields:

$$T_{\text{approximation}} = 2 + \frac{q}{2-4q} + \frac{4q}{2-4q} \quad (5.65)$$

To test the quality of this approximation, the expected delay in the two-station polite exhaustive ALOHA was studied by running a simulation program and measuring the expected delay for different arrival rates. The results of this simulation, and the approximation expressions suggested above, are plotted in figure 5.9. The three curves represent the light load expression, the heavy load expression and the approximation (equation (5.65)) expression. The black squares the simulation results. From this figure it is we see that the expression given in equation (5.65) serves as a good approximation of the expected delay in the system.

5.8.2 Noisy Exhaustive ALOHA: An Approximation

The analysis of the two-station noisy exhaustive ALOHA system, where the stations can not assume that they are the only stations in the system is trivial. This is implied from the following observation: When the number of stations in the system is exactly two, the "noisy" slot, following the end of a transmission period, will always be a successful transmission. This is true since at most one station will transmit at this slot. From this observation it can be concluded that the behavior of this system is identical to the behavior of the two-station system studied in section 5.5.

Therefore, we conclude that the expected delay in the two-station noisy exhaustive ALOHA system is given by equations (5.34) and (5.36) even if the two stations can not assume that they are the only stations in the system.

5.9 Summary

In this chapter we have derived the expected delay for several exhaustive slotted ALOHA schemes. In the case of two-station systems we have derived exact expressions; part of this analysis was done using the the queue with starter approach. For N-station systems a heavy-load approximation was introduced. This approximation was achieved by emulating the system by a random polling system. The approximation expression was shown to be very close to simulation

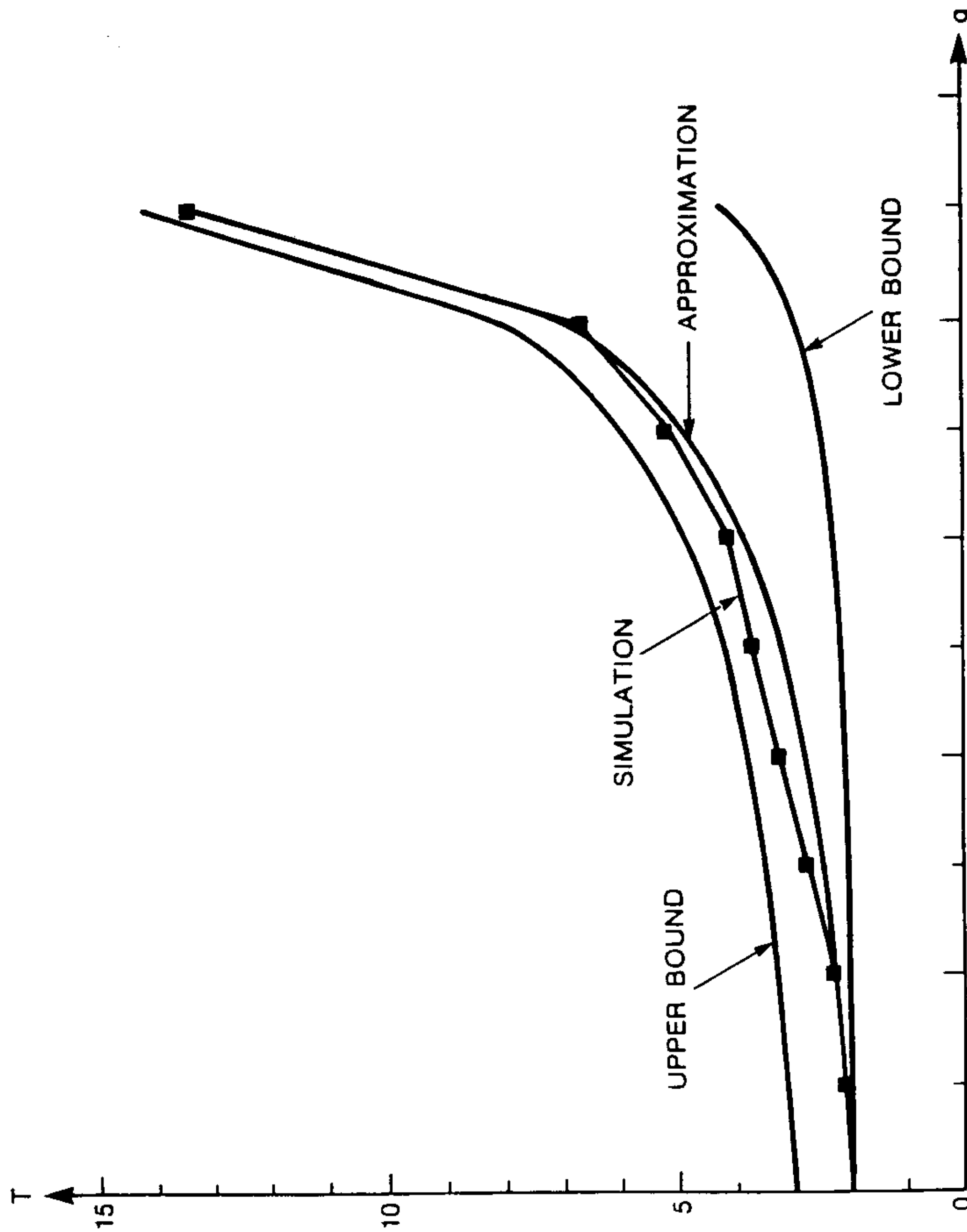

Figure 5.9: The expected delay in the two-station polite exhaustive ALOHA system

results. To approximate the expected delay for the whole range of load we introduced a heavy-load/low-load approximation. This approximation uses a heavy load approximation for the high load values, a low load approximation for the low load values, and a linear combination of these two approximations for the middle range.

CHAPTER 6

Synchronization Properties in the Behavior of a Slotted ALOHA Tandem

The tandem is one of the basic structures frequently used within communication networks. For this reason, understanding the behavior of a tandem is crucial for the invention of access and routing schemes, and in the performance analysis of multi-hop networks. This Chapter deals with the throughput analysis of a directional tandem in a multi-hop radio environment. The synchronization properties uncovered by this study suggest that the throughput achieved in this tandem is relatively high. It is shown that that the tandem throughput, due to the synchronization effects, is much higher than the value calculated for the throughput in models where "traditional" independence assumptions are used.

6.1 Introduction and Previous Work

Recent development and implementation of multi-hop packet radio networks has raised the need for the analysis of such networks. Compared to one-hop packet radio networks or to point-to-point wire networks these systems are very complicated and difficult to analyze.

A tandem is the simplest multi-hop packet radio network. Therefore, it is very attractive to study this system in order to get better understanding of general multi-hop packet-radio networks. Moreover, thinking about a general multi-hop network that carries the packets according to relatively stable routing, we realize that a general network is actually a collection of interconnected tandems. This is true since the route from one station to another can be considered as a directional tandem. For these two reasons we believe that studying the tandem behavior will contribute to the understanding of multi-hop networks. Discovering special properties related to tandems may lead to the invention of new access and routing schemes tailored for multi-hop networks. This is in contrast to existing schemes, that are based on the behavior of one-hop networks, and which have been modified to fit the multi-hop environments.

The behavior of a station located on a tandem is strongly correlated to the behavior of its neighbor station on the tandem. This is true since the packets one station transmits are received by its neighbor station. Therefore, it is expected that the events occurring in a tandem will be correlated to each other, and that commonly used independence assumptions may fail to predict the real behavior of the tandem.

For these reasons we choose to analyze in this chapter the behavior of a directional tandem under the Slotted ALOHA transmission policy.

The *one-hop* ALOHA system has been extensively studied in the past. For literature review of this area, see chapter 5 above.

One recent approach to study general *multi-hop* packet-radio networks was developed by Boorstyn and Kershenbaum [Boor80] , Sahin [Sahi82] and Tobagi and Brasio [Toba83]. This approach numerically solves for the set of attained throughputs as function of the set of offered traffic for given topology, routing and access schemes. The access schemes analyzed by this approach are the traditional one-hop access schemes when applied in a multi-hop environments. The drawback of this approach is that in order to analyze the system, strong "independence assumptions" have to be made which may hide important system properties.

A directional tandem network under a Slotted ALOHA policy was studied by Yemini [Yemi80]. The network model assumed by Yemini is depicted in Figure 6.1.

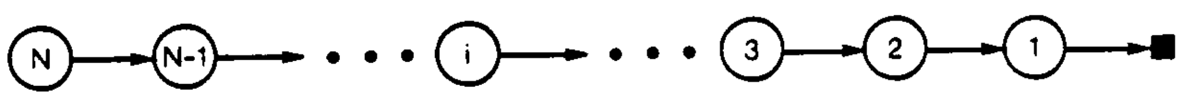

Figure 6.1: A tandem network

The assumptions made for this model are:

1. The time is slotted with slot size equals to the transmission time of a fixed size packet.

2. Station N is the only traffic source. The other stations do not generate any traffic.

3. All stations are under "heavy load", so that they always have something to transmit.

4. The access scheme used is Slotted ALOHA: At time slot t station i transmits with probability p_i. The transmission probabilities of different stations are not necessarily identical. The transmission of a station is independent of its actual buffer status due to the heavy load assumption.

5. The propagation delay is zero and acknowledgements are free and instantaneous.

6. A station in the system, let say i, only hears the packets transmitted by its neighbors, namely, stations $i+1$ and $i-1$ (This is with the exception that stations 1 and N hear only one neighbor).

7. Station i ($i=2,3,\cdots,N$) transmits its packets to station $i-1$. Station 1 transmits its packets to the destination.

The throughput S is defined as the rate of packets successfully leaving station N. Buffers are not considered in this model and are not "required" once assumption 3 is made. This assumption means that all stations are always under heavy load. A "fair" policy in this model is a set of transmission probabilities $\{p_i\}$ such that the probability of successful transmission from station i to station i+1 is identical for all stations.

Under this model Yemini analyzes the relation between the set $\{p_i\}$ and the system throughput. This is described in figure 6.2. For most practical networks (those which consist of more than four stations) it is shown that the maximal throughput is about 4/27.

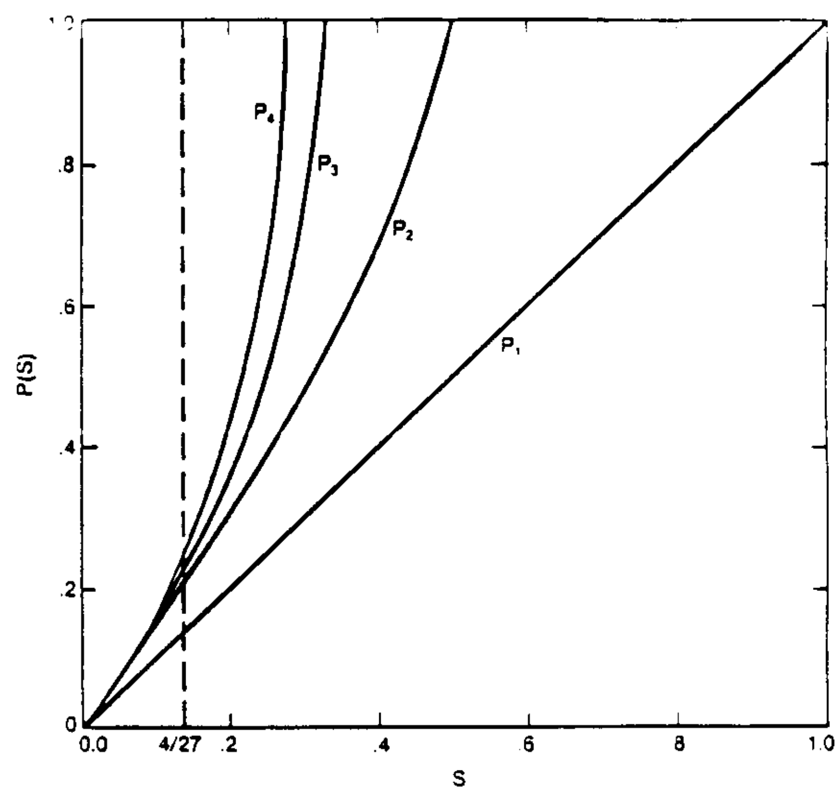

Figure 6.2: Tuned-up transmission policies from Yemini's model

An important observation made in [Yemi80] says that when all stations are "rude" (i.e: they all transmit with probability $p_i = 1$) the tandem becomes synchronized and the throughput is 1/3, much higher then the throughput achieved in figure 6.2!

The goal of this chapter is to study in more detail the behavior of the directional tandem under the slotted ALOHA access scheme. This is in light of the observations made in [Yemi80] with regard to the synchronization properties of this tandem. In section 6.3 we derive the tandem throughput under the assumption that the behavior of each station is independent of the behavior of the other stations. This analysis is based on the analysis done in [Yemi80]. In section 6.4 the independence assumption is relaxed and the tandem throughput is approximated both for heavily loaded system and for "relatively synchronized" system. Based on these two approximations a general approximation of the system throughput is then suggested. Lastly, section 6.5 discusses the properties discovered in sections 6.3 and 6.4.

6.2 Assumptions and Model Description

The model considered here is very similar to the one used in [Yemi80]. We consider a tandem of N stations, as described in figure 6.1. Station N is the source of all packets and the packets are destined to the *destination* station (the black station at the right hand side of the figure). Each station i ($i = 2,3,\cdots,N$) transmits its packets to station $i-1$ and station 1 transmits to the destination station.

The assumptions on the system (most of them identical to the assumptions made in [Yemi80]) are the following:

1. Time is slotted into equal length slots (each of them equal in duration to the transmission time of a fixed length packet). Without los of generality it is assumed that the length of a slot is a unit.

2. Station N is the only traffic source. The other stations do not generate any traffic.

3. The access scheme used is Slotted ALOHA: At any time slot t, every station flips a coin (with probability p for "heads"). If at time t station i has a packet in its buffer and if its coin shows "heads" then it transmits the packet; in all other cases station i keeps silent.

4. The buffer size of all stations is infinite.

5. The system is under heavy load. This means that station N always has something to transmit.

6. The transmission probability p is identical for all stations.

7. The propagation delay is zero and acknowledgements are free and instantaneous.

A transmission from station i to station $i+1$ is *successful* if at the transmission time both stations $i+1$ and $i+2$ are silent (equivalent special definition can be made with regard to the transmissions made by stations 1 and 2). The *throughput* of the system, S, is defined to be the expected number of packets successfully transmitted from station N to station $N-1$ per slot. It is clear that in this model, under equilibrium, this number is identical to the expected number of packets successfully received at any arbitrary station on the tandem, in particular, at the destination.

Using this model we are interested in finding the throughput S as function of the transmission probability p.

6.3 The Throughput Under Independence Assumption

In this section we derive the system throughput under the assumption that the behavior of every station is independent of the behavior of the other stations.

The following independence assumption decouples the behavior of station i from the behavior of the other stations:

INDEPENDENCE ASSUMPTION: The event "station i transmits at time t" is independent of the event "station j transmits at time t" for every $i \neq j$.

This assumption is very common in the analysis of shared channel networks, in particular in packet radio networks (see [Boor80, Sahi82, Toba83] for example). Clearly, this assumption is not true for the tandem system and the goal here is to test how close this assumption is to the real system.

Let T_i denote the probability that station i transmits a packet at time t. Note that T_i is different from p_i, since p_i is the probability that station i transmits at time t given that its buffer in not empty.

Under stability conditions, and under the assumptions made above, it is obvious that the tandem throughput equals to the probability that station i ($i=1,2,\cdots,N$) transmits a packet at time t, and this packet is successfully received at its destination (station $i-1$). Thus we have:

$$S = T_1 \tag{6.1a}$$

$$S = T_2(1 - T_1) \tag{6.1b}$$

$$S = T_i(1 - T_{i-1})(1 - T_{i-2}) \qquad ; \quad i = 3,4,\cdots,N \tag{6.1c}$$

The set of equations given by equation (6.1) can be solved to yield the throughput of the tandem. It is obvious that the model analyzed in this section is actually identical to the model studied in [Yemi80]. This is true, since the assumption that the transmission of station i at time t is independent of the transmission of station j is equivalent to the assumption that all stations are always busy.

This set of equations has been analyzed in [Yemi80] and the solution of this set is actually depicted in figure 6.2. The interpretation of this figure is the following: if the figure axis are interchanged, then the curve denoted by p_i represents the throughput in a tandem of length i as function of the transmission probability p. From this figure it is obvious that under the independence assumption the maximum achievable throughput for most tandems (those which consist of more than four stations) is about 4/27.

6.4 The Realistic Model: Discovering the Synchronization Properties

In this section we study the tandem behavior without using the independence assumption made in section 6.3 above. It is obvious that in order to give an exact analysis of the tandem one has to consider the state of all N queues (one in each station) in the system. This model is equivalent to the model of an N-dimensional random walk. Facing the fact that even a two-dimensional random walk is very difficult problem to solve (A comprehensive discussion about the problems related to the two-dimensional random walk can be found in [Yemi80] and an approach for solving such problems is suggested in [Cohe83]) we study this system using approximations.

6.4.1 A Heavy Load Approximation: A Lower Bound for the System Throughput

Our first approximation is a heavy load approximation. Let us assume that the system is heavily loaded, in particular, assume that the queues of stations $N-1$ and $N-2$ are never empty*. Under this assumption it is easy to calculate the system throughput from the expression for the probability that station N successfully transmits a packet to station $N-1$:

$$S = p(1-p)^2 \tag{6.2}$$

* Recall that from the system model station N is also always busy.

Considering the real system it is obvious that disregarding the transmission probability (p) used in the tandem, there exist always times at which either station N-1 or station N-2 (or both) are empty. Clearly, in these cases the probability that station N successfully transmits to station N-1 is higher than $p(1-p)^2$. Thus equation (6.2) forms a lower bound on the tandem throughput.

6.4.2 A Synchronized System: An Approximation of the Throughput at High Transmission Rates

In this sub-section we investigate the synchronization properties observed in [Yemi80].

First, let us consider, the *rude transmission policy.* According to this policy, as defined in [Yemi80], the transmission probability used by all stations is $p=1$. As observed in [Yemi80], if the system starts operating when all queues are empty, and if the rude transmission policy is used, then the system throughput is 1/3. This property is depicted in figure 6.3. In this figure the horizontal axis represents the tandem and the vertical axis represents time. An empty circle represents a station whose buffer is empty, and a bullet represents a non-empty buffer. Packet propagation (transmission) is represented by an arrow.

From figure 6.3 it is observed, that when the rude policy is used, the system is fully synchronized, and the throughput obtained by this policy (1/3) is the highest attainable throughput for a slotted ALOHA tandem.

Next, let us define more carefully the synchronization property observed above: A station, say i ($i=1,2,\cdots,N-1$), on the tandem is called a *type 1 station* if $i(\text{mod } 3)=1$. Station i is called a *type 2 station* if $i(\text{mod } 3)=2$. Station i is called a *type 3 station* if $i(\text{mod } 3)=0$. According to this definition the stations of type 1 are: $1,4,7,\cdots$; the stations of type 2 are: $2,5,8,\cdots$; and the stations of type 3 are: $3,6,9,\cdots$. A tandem of N stations is *fully synchronized at time t* if at this time one of the following holds:

1. All stations of type 1 have exactly one packet in their buffer, and all the rest of the stations have empty buffers.

2. All stations of type 2 have exactly one packet in their buffer, and all the rest of the stations have empty buffers.

3. All stations of type 3 have exactly one packet in their buffer, and all the rest of the stations have empty buffers.

It is obvious, from this definition, that the tandem depicted in figure 6.3 is fully synchronized at every time $t \geq 5$.

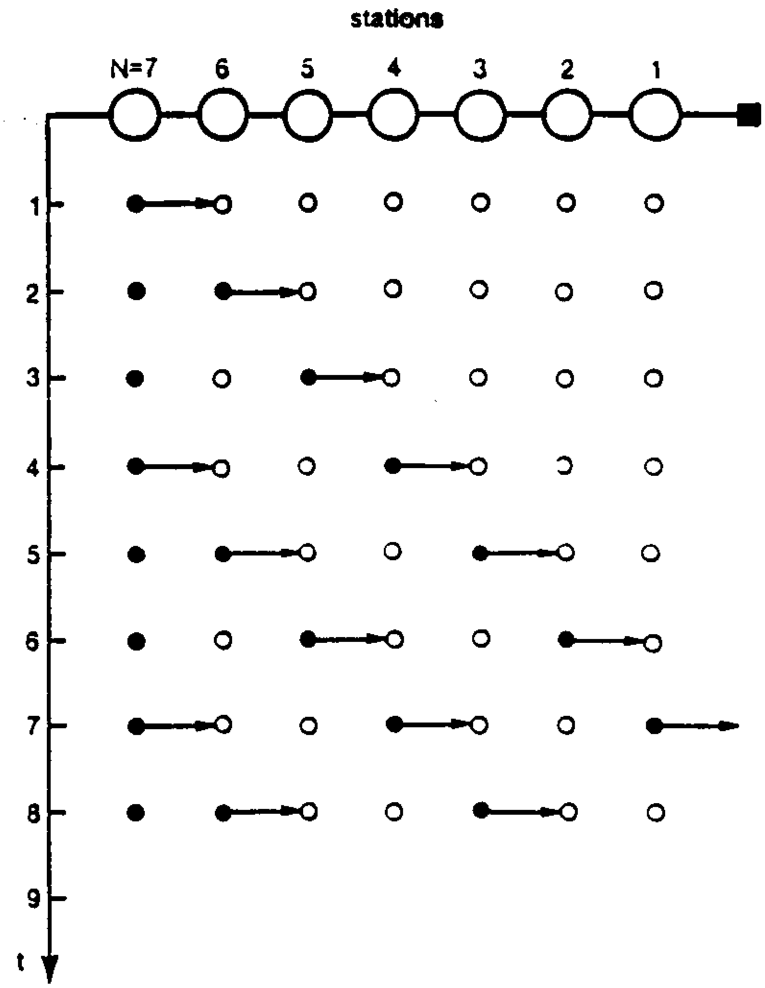

Figure 6.3: Packet propagation in a seven station rude tandem

The observation made in [Yemi80] actually states that if the system starts operating when all the queues are empty and if the rude policy is used then after at most $N-1$ time units the tandem is fully synchronized and its throughput is 1/3.

A similar observation can be made, even if not all the queues are empty at the starting time:

THEOREM 6.1: Let $m_1, m_2, \cdots, m_{N-1}$ be the number of packets found at time 0 in queues $1,2,\cdots,N-1$, respectively. Let $m = \sum_{i=1}^{N-1} m_i$. Let us assume that the system starts operating at time 0, using the rude policy. Then at every time $t > \max(N, m/3)$ the tandem is fully synchronized, and the system throughput is 1/3.

Theorem 6.1 can be proved by a simple induction, a proof which is avoided here for its simplicity.

The importance of theorem 6.1 is that independently of the system state, if at time t all stations start using the rude policy, the tandem will eventually (and relatively fast!) become fully synchronized.

After understanding the tandem behavior under the rude policy, we next discuss the system performance when $p \neq 1$. The goal here is to derive the system throughput when p is relatively large. This will be done by assuming that the system is fully synchronized, and by observing its behavior when $p = 1 - \epsilon$ and $\epsilon \approx 0$.

Let us assume that the tandem is fully synchronized and that the transmission probability used is $p = 1 - \epsilon$. Under these assumptions it is clear that the normal operation of the tandem is the following: At time t all stations whose buffer is not empty (successfully) transmit their packet to their down stream neighbor. An exception to this "normal" operation occurs when one of the stations, say i, whose buffer is not empty, does not transmit its packet. This can happen since the transmission probability is $p < 1$. In this case we say that a *failure occurs at station i at time t*. The point (i,t) in the time space domain, is called, in this case *a failure point*.

Let us assume that at time $t-1$ the system is fully synchronized, and that at time t a failure occurs at station i. In addition, let us assume that no other failure occurs in the system at time t or afterwards. In figure 6.4 we depict a fully synchronized tandem consisting of 16 stations, where a single failure occurs at station 9 at time 5. The points in this figure represent the location of the packets on the tandem. A bullet in this figure represents a successful transmission, and an empty square or a cross represents an unsuccessful transmission. A cross in this figure (point (9,5)) represents a failure point; a point where a station whose buffer is not empty chooses not to transmit (with probability $1-p$). An empty square represents a different type of unsuccessful transmission: in this case a packet is transmitted but, due to noise, it is not successfully received by its destination station. For the simplicity of the figure the empty-queue stations are not explicitly depicted in this figure.

From figure 6.4 we make the following observations on a fully synchronized system:

1. In the time space domain, a fully synchronized system is represented by a uniform structure of diagonals, going from top left to bottom right. Each of these diagonals represents the propagation of one packet over the tandem. These diagonals are three units apart from each other.

2. A single failure occurring at station i at time t affects the uniform structure by producing two *waves* as following:

   a. A *forward wave,* which is a diagonal that propagates from the failure point to the bottom-right direction. Along this wave the distance between two consecutive packets is four units, instead of normally three units.

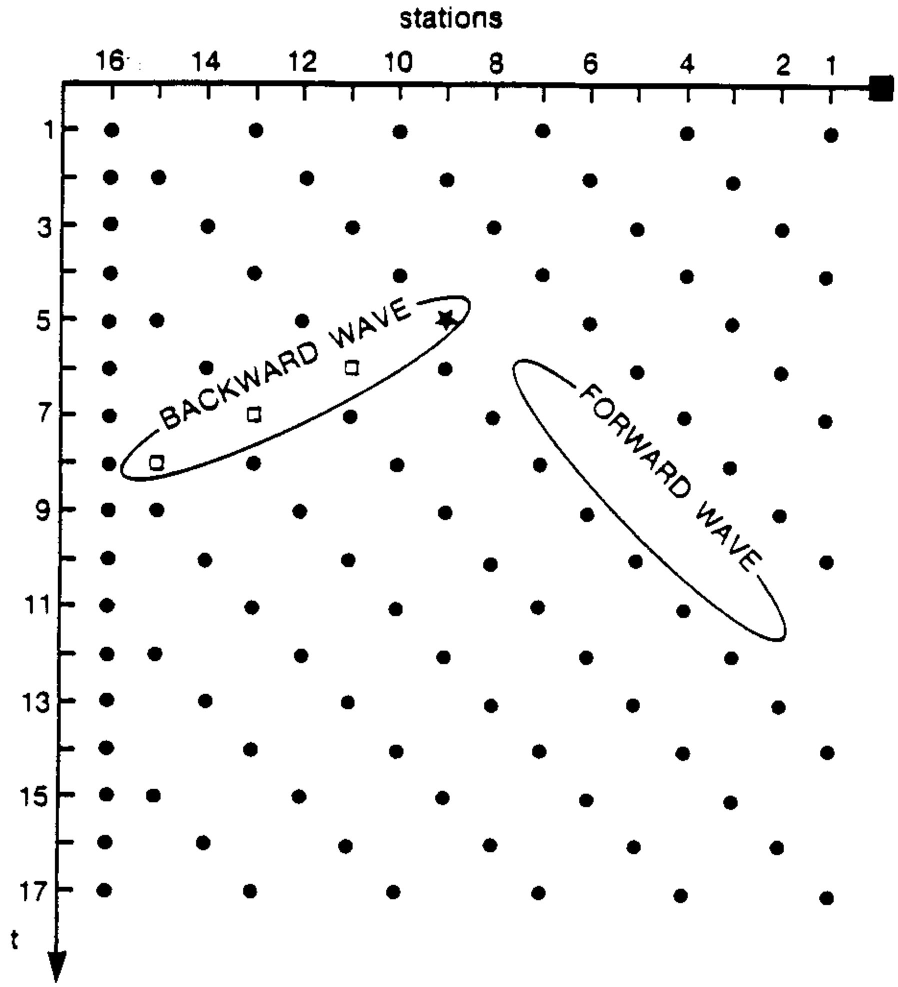

Figure 6.4: A failure in a fully synchronized tandem

b. A *backward wave*, which is a diagonal that propagates from the failure point to the bottom-left direction. This wave contains all the unsuccessful transmissions occurring in the system as a result of the failure occurring at station i at time t.

To calculate the tandem throughput in the presence of failures let us observe the system behavior at the destination. From figure 6.4 we see that when the tandem is fully synchronized packets arrive at the destination at rate of one packet per three time units. When a single failure occurs, the interarrival time between the failed packet to the packet preceding it is four time units. From this observation it is easy to calculate the throughput when single failures occur in the system. Let us call a packet who arrives to the destination four units of time behind the packet preceding it a *delayed packet*. Let F be the *failure rate* as observed by the destination; i.e., this is the expected number of diagonals who get shifted by a forward wave, per unit of time. Let a *cycle* be the period starting right after the arrival of a delayed packet to the destination and ending right after the arrival of the preceding delayed packet to the destination. Let x be the number of packets arriving to the destination in some cycle.

From these definitions, we see that $x-1$ of the interarrival times observed by the destination during the cycle are of length 3, and one of these interarrival times is of length 4. Thus the length of the cycle is $3x+1$ time units. The expected length of the cycle can be calculated from the failure rate:

$$E[\text{cycle length}] = \frac{1}{F}+1 \tag{6.3}$$

This is true since for every failure occurring in the system, one packet gets delayed by additional unit of time. Similarly the expected number of packets arriving to the destination in a cycle is thus:

$$E[\text{ number of packets arriving at the destination during a cycle}] = \frac{1}{3F} \tag{6.4}$$

Thus, the system throughput can be calculated from equations (6.3) and (6.4):

$$S = \frac{\frac{1}{3F}}{\frac{1}{F}+1} = \frac{1}{3(1+F)} \tag{6.5}$$

Now, let us calculate F, the failure rate. Before calculating F, we should note that not all failures occurring in the system are sensed by the destination. The reason is that some of the failures may "cancel" each other. In particular, the next theorem deals with failures which occur concurrently.

THEOREM 6.2: Let (i,t) and (j,t) be two failure points in the time space domain and let $i<j$. Then the forward wave produced by (i,t) is not sensed by the destination.

We avoid a formal proof of this theorem; the reader may convince himself by observing figure 6.5 where two concurrent failures are depicted. Each of the two failures, one occurring at station 7 at time 8 and the other occurring at station 16 at time 8, generates a forward wave and a backward wave. However, the forward wave generated at the point (16,8) and the backward wave generated at the point (7,8) "cancel" each other. Thus, only one forward wave, the one generated by station 7, arrives at the destination.

From theorem 6.2 we may conclude that if k failures occur in the tandem at time t at most one of them will be sensed in the destination. Thus, at most one of these concurrent failures should be counted in the throughput calculation.

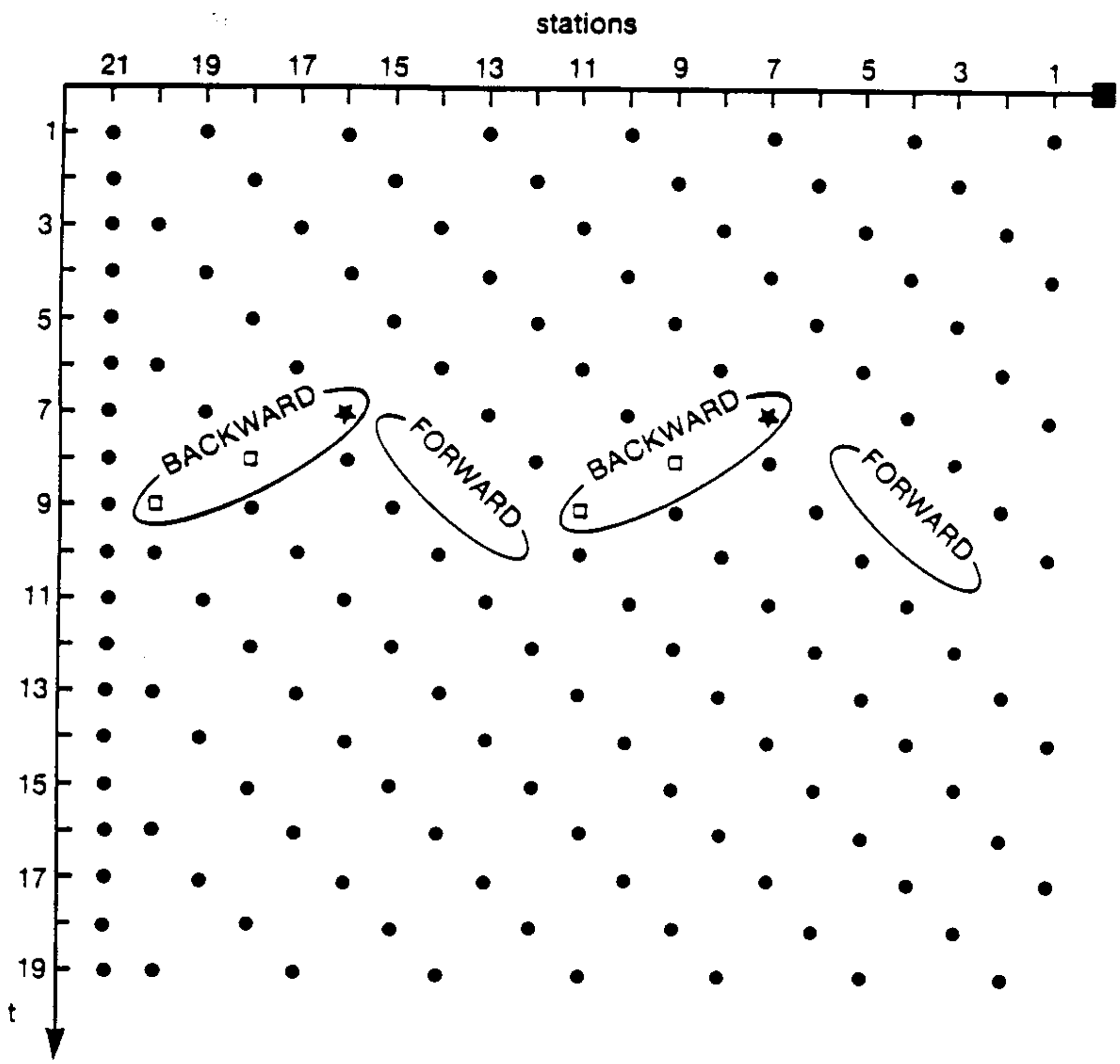

Figure 6.5: Two concurrent failures in a fully synchronized tandem

A more complicated task is to consider non-concurrent failures. It can easily shown that not only concurrent failures, but also failures who occur at different times may "cancel" each other. For example observe figure 6.5 and suppose that in addition to the failures depicted in the figure, a failure also occurs at time 6 at station 20. The forward wave generated from this failure will get canceled by the backward wave generated from the failure (16,7).

The problem with non-concurrent failures is that it is quite complicated to estimate their effect. For this reason we simplify our calculation by the following assumption:

ASSUMPTION: The waves produced by non-concurrent failures do not cancel each other.

Under this assumption and from theorem 6.2 we can now calculate F, the failure rate as observed by the destination. This is simply the probability that at least one failure occurs at time t in the whole tandem. This is true since for every set of concurrent failures we count exactly one failure. Since in the average, the number of stations whose buffer is not empty, is $N/3$ (recall that the system is synchronized) then the probability that at least one failure occurs at time t can be estimated by:

$$F = 1 - p^{N/3} \tag{6.6}$$

From equations (6.5) and (6.6) we finally calculate the tandem throughput as a function of the transmission probability p:

$$S = \frac{1}{3 \cdot \left(2 - p^{N/3}\right)} \tag{6.7}$$

Note that the expression given in equation (6.7) should be good only for relatively high values of p, since in the derivation of (6.7) we assumed that the system is fully synchronized. Note also that this expression should give a lower bound on the tandem throughput; the reason for this is that wave cancellations due to non-concurrent failures are not considered in this calculation.

6.4.3 A General Throughput Approximation

Two expressions have been suggested above for the tandem throughput: 1) A heavy load approximation given by equation (6.2) 2) A high transmission-rate approximation given by equation (6.7). To test the quality of these approximations we compared them to simulation results. In figure (6.6) we plot the system throughput as a function of the transmission probability in a 4 station system ($N=4$). The continuous curve in this figure represent simulation results and the dashed curves represent the approximations. In a similar way, figures (6.7) and (6.8) depict the tandem throughput in a nine station system ($N=9$) and in a nineteen station system ($N=19$), respectively.

The following observations can be made on the quality of the approximations:

1. The heavy load approximation, given by equation (6.2) is very good for the lower range of transmission probability. The quality of this approximation increases with the size of the tandem; the reason for this property is that the longer the tandem is, the more the

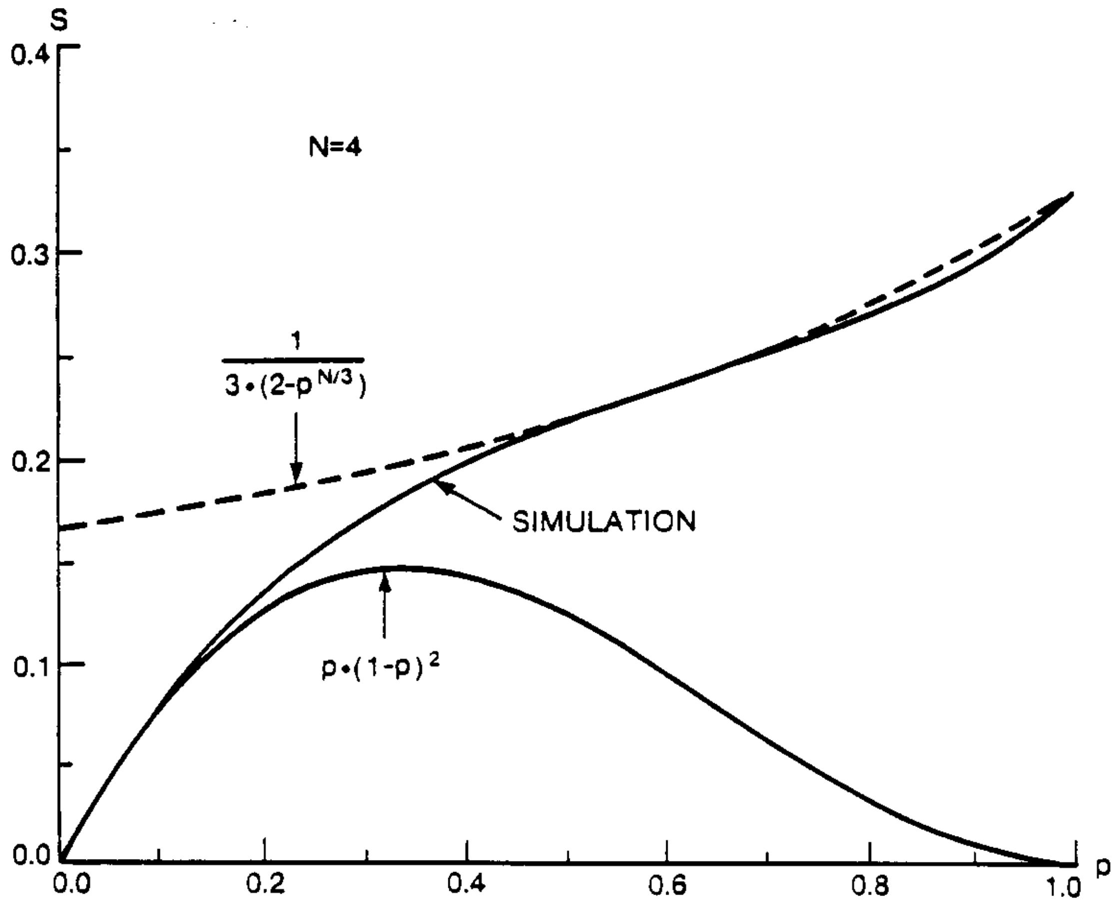

Figure 6.6: The throughput in a 4-station tandem

stations are likely to be heavily loaded.

2. The high transmission rate approximation is good, as expected, in the higher range of the transmission probability. The quality of this approximation decreases with the tandem size; The reason is that in long tandems the likelihood that two non-concurrent waves will cancel each other is higher than in short tandems. Since this effect is neglected in equation (6.7), the approximation for short tandems is better than the one for long tandems. Nevertheless, it is observed that for most practical purposes (tandems which are shorter than 20 stations), this approximation is relatively good.

Using the two approximations suggested above we next suggest a general approximation of the system throughput. This approximation can be constructed as following:

1. For the low range of p (p smaller than the point p_1, which is to be defined below) use the heavy load approximation given by equation (6.2).

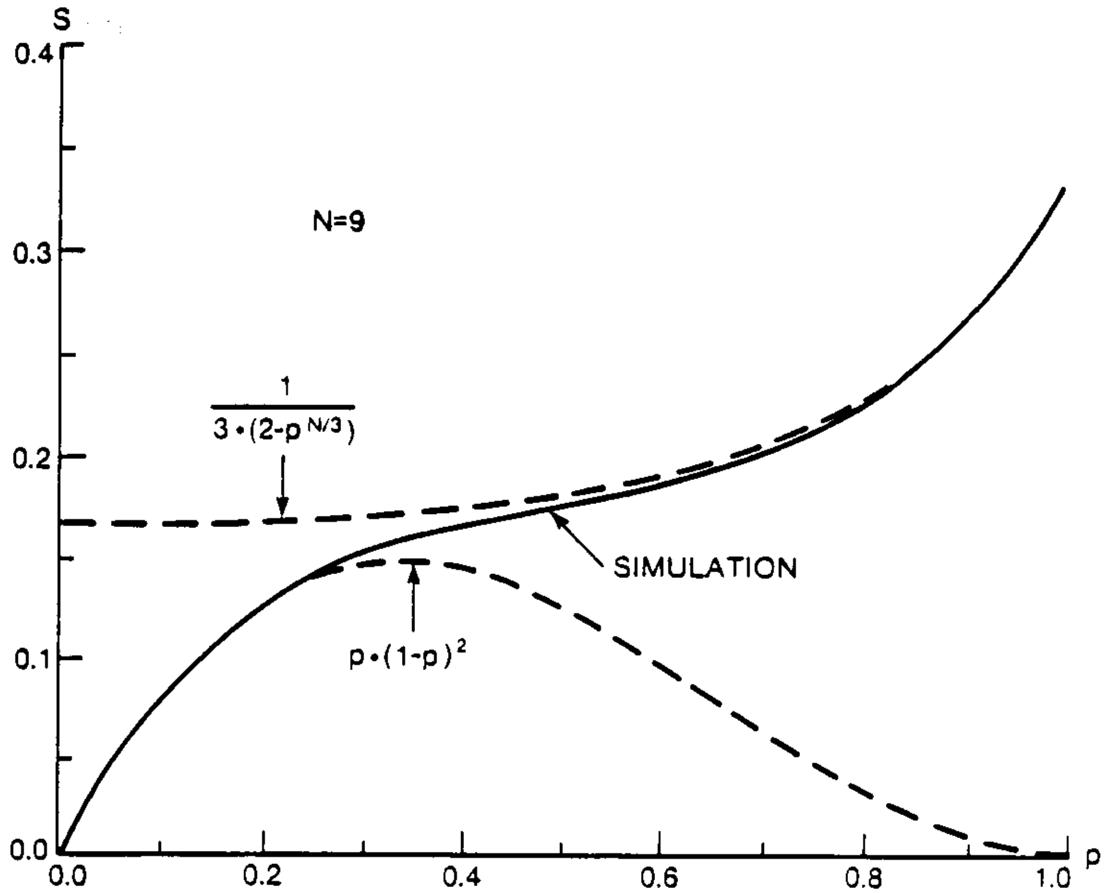

Figure 6.7: The throughput in a 9-station tandem

2. For the high range of p (p greater than the point p_2, which is to be defined below), use the high transmission-rate approximation given by equation (6.7).

3. For the middle range ($p_1 \leq p \leq p_2$) use a linear approximation, $S = a \cdot p + b$, tangent to the curves given by equations (6.2) and (6.7).

4. The four parameters required for this approximation, a, b, p_1 and p_2 can be determined by solving a simple set of equations, consisting of equations (6.2), and (6.7), and of the linearity and tangency constraints. This set is:

$$a \cdot p_1 + b = p_1 \cdot (1 - p_1)^2 \tag{6.8a}$$

$$a = (1 - p_1) \cdot (1 - 3p_1) \tag{6.8b}$$

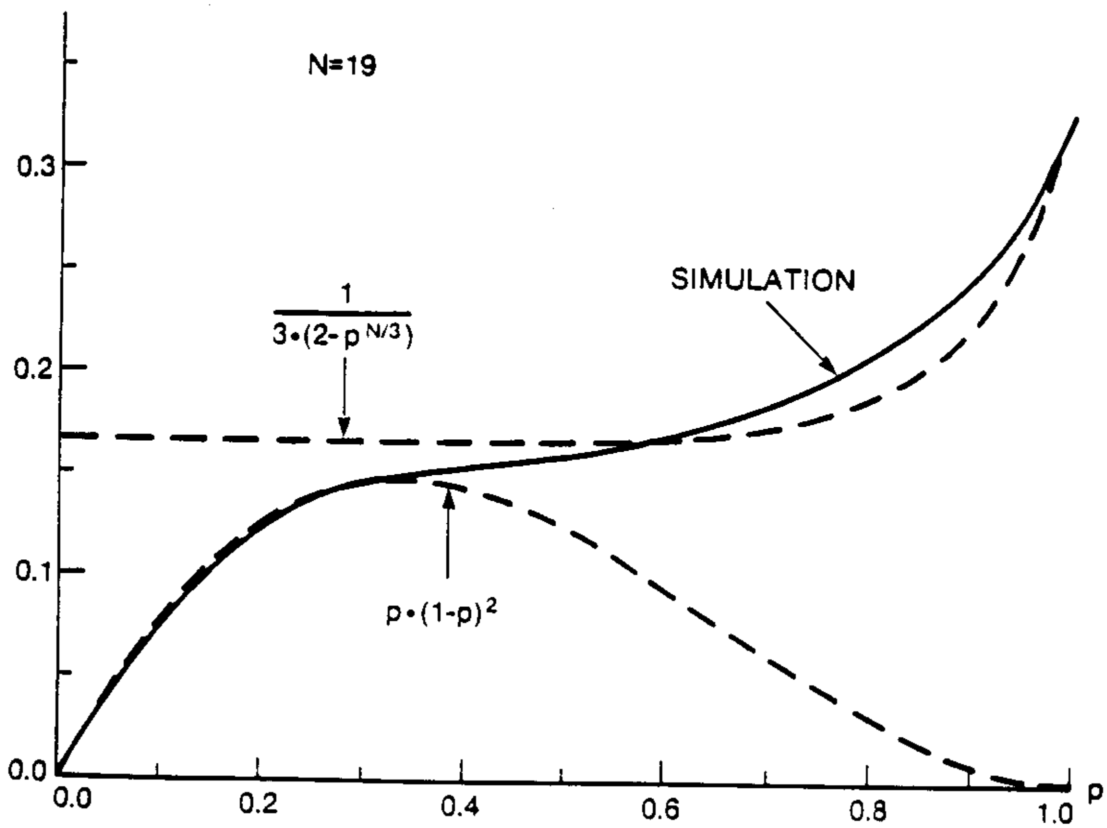

Figure 6.8: The throughput in a 19-station tandem

$$a \cdot p_2 + b = \frac{1}{3 \cdot \left(2 - p_2^{N/3}\right)} \tag{6.8c}$$

$$a = \frac{N p_2^{N/3-1}}{9 \cdot \left(2 - p_2^{N/3}\right)^2} \tag{6.8d}$$

6.5 Discussion

Two types of behavior have been observed for the slotted ALOHA directional tandem:

1. When the transmission probability p is small, the system behaves in non-synchronized way. Under this policy the queues in the system are usually non-empty and the tandem

can be considered as heavily loaded. Thus, the behavior of each station is independent of the behavior of the other stations.

2. When the transmission probability is high the system gets synchronized. It is evident that the transmissions are usually phased in waves, and thus the number of collisions is relatively low and the throughput is high.

In contrast to what could be thought from [Yemi80] the system moves from a heavy-load behavior to a full-synchronization behavior in a continuous way: the higher the transmission probability the more likely the tandem to be synchronized.

It is observed that the throughput in the system monotonically increases with the transmission probability p. This property suggests that high transmission probabilities can be used very efficiently in multi-hop radio networks to yield high throughput over tandems. It should be emphasized that $p = 1$ cannot be used safely in general multi-hop networks since the use of rude policy in such networks may cause eternal deadlocks between conflicting stations.

The synchronization behavior suggests that using independence assumptions in the analysis of multi-hop radio networks may lead to a wrong analysis of the system behavior. It is evident that the results derived by using independence assumptions (see figure 6.2 and the heavy load approximation in figures 6.6, 6.7 and 6.8) may be absolutely wrong in predicting the system behavior. Moreover, using those results in the invention of access schemes and in the process of tunning up their parameters, may cause these schemes to behave inefficiently.

CHAPTER 7

On the Behavior of a Very Fast Bidirectional Bus Network

In this chapter we study the behavior of the very fast bidirectional bus system. The bidirectional bus system has been investigated in the past under the main assumption that the propagation delay incurred by a packet is relatively small in comparison to its transmission time. Under this assumption, it has been shown that if the packet transmission time decreases the performance of existing access schemes (like CSMA) degrades. Recent technological developments (such as fiber optics) in communication networks have brought up new faster bus networks. For these networks it cannot be assumed any more that the propagation delay is relatively small in comparison to the transmission time. This chapter deals with analyzing the very fast bidirectional bus system. In contrast to the previous studies, the assumption that the bus is very fast is inherently embedded in the system model. The results derived in this chapter show that due to self synchronization properties observed in the system at high loads, the system performance is not poor as implied from previous studies.

7.1 Introduction and Previous Work

In a local area network a channel is shared among many stations which are (relatively) close to each other. One of the common topologies for such a network is the bidirectional bus (like Ethernet) and one of the most popular access schemes for this topology is the Carrier Sense Multiple Access (CSMA). In this scheme a station behaves in a "polite" way: when it wants to transmit, it first senses the channel. If the channel is found to be idle the station will transmit the packet, and if the channel is busy the station remains silent and postpones its transmission attempt. An improvement of CSMA is CSMA with Collision Detection (CSMA-CD). In this scheme, in addition to the protocol described above, a station can detect if it is involved in a collision. If a collision is detected the station aborts its transmission and repeats the scheme described above.

Both access schemes take advantage of the very short propagation delay (relative to the transmission time). The ratio between the propagation delay and the transmission time is denoted by a and can be thought of as the number of packets "contained in" the bus.

The performance of CSMA was studied by Kleinrock and Tobagi in [Klei74, Klei75, Toba74]. The performance of CSMA-CD was studied by Tobagi and Hunt [Toba79].

The following properties are observed with respect to these access schemes:

1. Both schemes are superior to Aloha and Slotted-Aloha. The superiority is in terms of higher throughput and lower delay.

2. The attained throughput, S, of both systems, increases with the offered load, G, until it reaches its maximum. After this point (very high load) the throughput decreases. This property (and the properties described above) is shown in figure 7.1 (taken from [Klei76]).

3. The maximum attainable throughput, denoted by the system capacity, decreases with a. This is shown in figure 7.2 (taken from[Klei76]). It is observed that the performance of these schemes is good as long as $a \leq .05$.

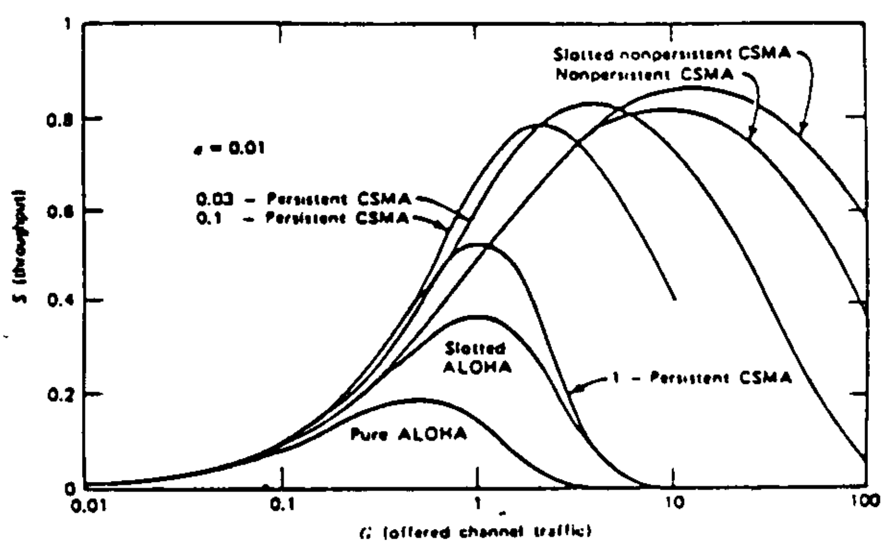

Figure 7.1: Throughput for the various random access modes (a=0.01)

Technological developments (such as fiber optics) in communication networks have recently increased the speed of the communication channel, and future developments are likely to increase it even more. Other technological improvements allow the future networks to be longer and longer. These trends lead the communication industry to the building of systems

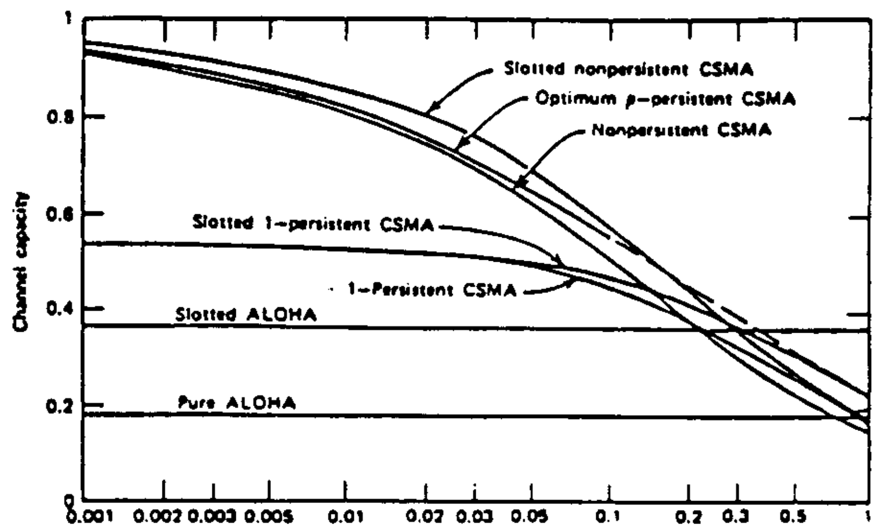

Figure 7.2: Effect of propagation delay on channel capacity

where the parameter *a* is larger and larger. Observing the properties mentioned above, we realize that CSMA and CSMA-CD *may* not be efficient in these future systems. Nevertheless, it is not really clear what will be the performance of these access schemes under the presence of *big-a*. This is so, since all the properties described above were derived from models that *assumed* relatively small *a* (very small propagation delay). It is therefore desirable to study the behavior of a shared-bus communication system under the assumption of *big-a*.

The behavior of such systems has been recently investigated by several studies. However, these studies concentrated on suggesting semi-organized access schemes for these networks and not on studying the behavior of these networks under the CSMA scheme. The main principle of these schemes is to organize the packets transmitted in the system to efficiently use the channel. These studies are reported in [Frat81, Gerl83a, Gerl83b, Limb82].

This chapter is devoted to the study of multi-access bus systems under the assumption of *big-a*. The purpose here is to take, as a basic assumption, the fact that the communication channel is fast and that the parameter *a* is large. For this type of system we study its behavior and analyze its performance when controlled by different access schemes.

Two main aspects of this system are studied in this chapter. First, in section 7.3 we study the theoretical limitations of the very-fast shared bus system. The main goal in this section is to calculate the maximum throughput which can be achieved in the system, neglecting the randomized behavior of the system inputs. The capacity of the system, defined to be the highest attainable throughput, is derived in this section, under several assumptions. The main result of this section is that the capacity of the very fast bus system is about 2. This means that it is possible to schedule packet transmissions in the system such that the expected number of packets transmitted and successfully received (per slot*) is about two.

Second, in section 7.4 we investigate the system behavior under the assumption of stochastic arrivals. The model used in this section is similar to the models used in the analysis of slotted ALOHA and CSMA-CD; however, in contrast to those models, this model encaptures the correlation between events occurring in the system. This property is rather important since the correlation between events in the very fast bus system seriously affects the system performance. The main property discovered in this analysis is that in contrast to previously studied shared channel systems, this system is very stable and the system throughput always increases with the offered load.

7.2 Model Description

The system considered here consists of N stations connected by a bidirectional bus. It is assumed that the stations are located on the bus such that the distance between every two neighboring stations is exactly a unit. The stations are numbered $1,2,\cdots,N$ from left to right. Since the bus is bidirectional a transmission which is originated at station i will propagate in both directions on the bus.

The time is slotted with slot size equals to the propagation delay a transmission incurs when propagating between two neighboring stations. This means that if station i starts transmitting at time t, then station $i+1$ will start hearing this transmission at time $t+1$. The time interval, starting at time t and ending at time $t+1$, is called the tth slot. Every packet transmission starts at the beginning of some slot.

The transmission media is assumed to be very fast, such that the length of a fixed size packet, measured in terms of distance, is smaller than or equal to the distance between two neighboring stations. This implies that the parameter a of this system is $a \geq N-1$. For simplicity we assume that the packet size exactly equals to the distance between neighboring stations, i.e., $a = N-1$.

---

* The duration of each slot is the transmission time of a packet.

Due to the above assumptions, the traditional model which considers only the timing of events can not be used to model our system. The reason is that each event must be represented by two parameters: time and location. For example, consider that station 1 starts transmitting a packet at time t. Station 2 will start hearing this packet at time $t+1$, station 3 will start hearing this packet at time $t+2$ and so on.

To represent the system behavior, we use the time-space domain. In this domain the horizontal axis is used to represent the bus (on which the stations $1,2,\cdots,N$ are located left to right) and the vertical axis represents the time. The propagation of a packet, in this domain, is represented by a band. For example, see figure 7.3 which depicts the propagation of a packet, transmitted at slot t from station 2.

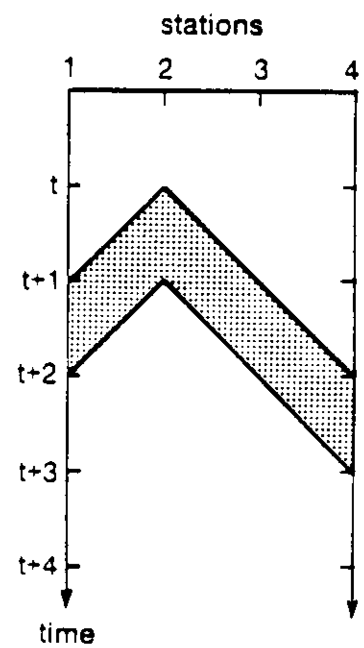

Figure 7.3: The Representation of a Packet in the Time-Space Domain

This packet is heard by stations 1 and 3 at slot $t+1$ and by station 4 at slot $t+2$.

In contrast to the traditional model, packets which collide, are not assumed to destroy each other. Rather, they are assumed to "pass through" each other. For example, consider the two packets depicted in figure 7.4. These two packets are transmitted concurrently at slot t by stations 2 and 4. At slot $t+1$ the packets collide at station 3 and thus non of them is heard properly by station 3. However, the packets pass through each other, so at slot $t+2$ one of them is heard correctly by station 2 and the other one is heard correctly by station 4.

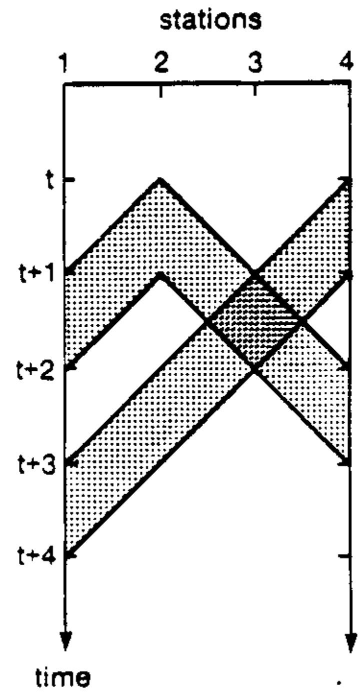

Figure 7.4: Two Packets Pass Through Each Other

From the above description, it is implied that the terms: *idle slot, successful slot* and *collision slot* are not global properties of the system but, rather, local properties of a given station. Slot t may be a collision for station i and idle for station j. Similarly, a given packet may be successfully heard by station i and unsuccessfully heard by station j.

A time slot t is said to be *idle at station* i if no transmission is heard by station i during this slot. Slot t is said to be *successful at station* i if exactly one transmission is heard by station i during this slot. Slot t is said to be *collision at station* i if more than one transmissions are heard by station i during this slot. A packet x is said to be *heard correctly by station* i *at slot* t if x is heard by station i at slot t and no other packet is heard by i during this slot. A packet x is said to be *collided at station* i *at slot* t if x is heard by station i at slot t and and x is not the only packet heard by station i at this slot.

To analyze the system performance one has to recognize for every slot t and for every station i if t is a successful slot at station i or if it is an idle (collision) slot at this station. From the assumptions made above, it is clear that a given packet is heard by station i during slot t if and only if, the head of the packet is heard by station i at time t. Therefore, to analyze the system, one can simplify the representation of packets: instead of using a band to represent the transmission of the packet, use a line to represent the time at which the packet head is heard.

According to this simplification, figure 7.4 is transformed into figure 7.5.

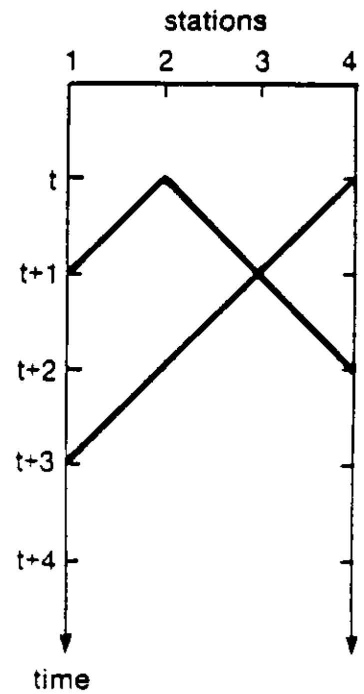

Figure 7.5: A Simplifying Representation of the Packets Propagation

When analyzing the performance of a shared channel communication media one has to be careful in defining the performance criteria. A shared channel communication network can be used both for broadcast and for point-to-point communication purposes. A *broadcast transmission (packet)* is a transmission (packet) which is originated at station i and has to be received correctly by all other stations. A *point-to-point transmission (packet)* is a transmission (packet) which is originated at station i, destined to station j ($j \neq i$) and has to be received correctly at station j.

In this chapter our main interest is in the point-to-point performance of the system. Thus, it is assumed that each packet x is destined to a specific station, let say i and has to be successfully heard by that station. The way by which this packet is heard by the other stations (successfully or not successfully) is not relevant to this analysis. Following this assumption we have to define the successful reception of a packet more carefully: A packet x is said to be *suc-*

cessfully received[*] *by station i at slot t* if x is destined to station i and if it is successfully heard by i at slot t.

In this chapter we will study the behavior of the system under different *transmission policies*. A *transmission policy* is a set of rules, used by the stations in the system to determine when a station can transmit and when it stays silent.

As stated in the introduction, the influence of two general transmission rules, the *politeness* rule and the *fairness* rule, will be examined in this chapter. These rules are defined next.

A station is said to be *polite* if it does not transmit when it hears transmission originated from another station. A station is said to be *polite to the left* if it does not transmit when it hears transmission originated at a lower index station (i.e., transmission that arrives from the left, according to our representation). A station is said to be *polite to the right* if it does not transmit when it hears transmission originated at a higher index station. A transmission policy is said to be *bidirectional-polite policy* if all stations in the system are polite. A transmission policy is said to be *unidirectional-polite policy* if every station in the system is either polite to the left to polite to the right. A transmission policy is said to be *left-polite policy (right-polite policy)* if every station in the system is polite to the left (polite to the right)

A transmission policy is called a *fair policy* if for every to stations i and j, station j is allowed to transmit between any two consecutive transmissions of station i. A transmission policy is called a *strictly fair policy* if for every four stations i, j, k, and l, station i is allowed to transmit to station j between any two consecutive transmissions from station k to station l.

To better understand the politeness rule, let us observe the transmissions depicted in figure 7.5., and let us assume that these are the only transmissions in the system. Under polite policy, station 1 is not allowed to transmit at times $t+1$ and $t+3$; station 2 is not allowed to transmit at time $t+2$; station 3 is not allowed to transmit at time $t+1$ and station 4 is not allowed to transmit at time $t+2$. Under left-polite policy stations 1 and 2 have no transmission constraints; station 3 is prohibited from transmitting at time $t+1$ and station 4 is prohibited from transmitting at time $t+2$.

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[*] In contrast to the previous definition of "successfully heard".

7.3 On the Capacity of the System

In this section we study the capacity of a very fast bidirectional bus system. The importance of this study is that the capacity of a system forms a reachable upper bound on the system transmission capabilities, and thus provides an evident for the system potential.

7.3.1 The Definition of Capacity

In the following we are interested in the "logical" capacity of the system. For a given system, consisting of N stations, it is assumed that a station can transmit one unit of information (one packet) per one unit of time. Since our interest is in the point-to-point performance of the system, a packet transmission is defined to be successful if the packet is transmitted by station i, destined to station j, and received correctly by station j. Let t denote the time, and let $P(t)$ denote the number of packets transmitted and successfully received in the system by time t. The *throughput* of the system is denoted by S and defined as:

$$S \triangleq \lim_{t \to \infty} \frac{P(t)}{t}$$

The *capacity* of the system, denoted by C, is defined to be the highest achievable throughput of the system.

To understand the capacity definition and its importance, let us calculate the capacity of some simple systems.

First, let us consider a system consisting of two stations, connected by a point to point line. Each of the stations can either transmit or receive, but cannot transmit and receive simultaneously. This system is depicted in figure 7.6.

Figure 7.6: The Capacity of a Point to Point Two Station System is 1

It is obvious that the capacity of this system, according to the definition given above, is 1; i.e., the maximal number of packets that can be transmitted and received, per unit of time, is 1. It is also obvious that if in this system, the stations are able to transmit and receive concurrently, than the system capacity is 2.

Next, let us consider a system which consists of N stations where each station is connected to every other station by a point to point line. Like in the previous system, here too, each station can either transmit or receive, but cannot do both concurrently. This system, is depicted in figure 7.7.

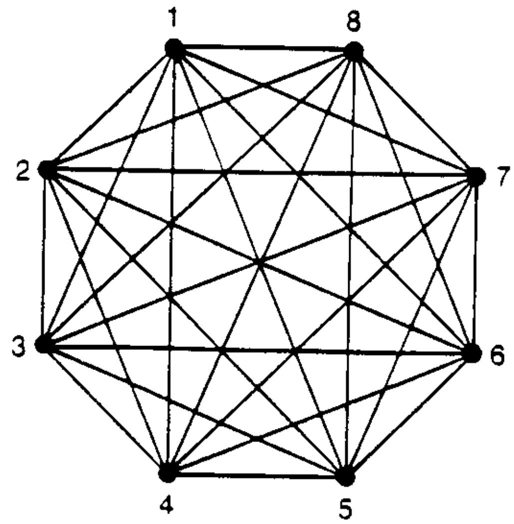

Figure 7.7: The Capacity of a Point to Point Fully Connected N Station System is $N/2$

It is easy to see that the capacity of this system, if N is even, is $N/2$. The reason is that the maximum number of conversations that can be handled concurrently, is $N/2$.

The last system to be considered is the one hop packet radio network consisting of N stations. Here, if the distance between the stations is assumed* to be negligible, then at every moment at most one station can transmit. The reason is that if two stations transmit concurrently the packets collide and get garbled. Therefore, the point-to-point capacity of this system is 1.

It should be noted that the definition of capacity as given above, gives a measure for the *concurrency* of the system. The capacity of the system can be thought of as an average over time of the maximum number of conversations that can be held concurrently in the system.

* This is the traditional assumption in the analysis of one hop packet radio networks

Having defined the capacity measure, we now calculate the capacity of our system under several constraints. First, the capacity of an unconstrained system is derived.

7.3.2 Two Upper Bounds on the System Capacity

In the following, we derive two upper bounds for the throughput in the system. These will serve also as upper bounds on the system capacity. Before deriving these bounds, some more definitions related to the time-space domain are required.

A point (t,i) in the time-space domain is called a *transmission point* is station i transmits a packet at slot t (i.e., starts transmitting at time t). A point (t,i) in the time-space domain is called a *reception point* is station i hears a single transmission at slot t and if this transmission is destined to station i. A line which contains the points $(t,1)$, $(t+1,2)$, $(t+2,3)$, $\cdots$, $(t+N-1,N)$ is called a *left diagonal* (a diagonal that starts from top left and goes to bottom right). Similarly, a line which contains the points (t,N), $(t+1,N-1)$, $(t+2,N-2)$, $\cdots$, $(t+N-1,1)$ is called a *right diagonal.*

Using this notation we next derive the upper bounds on the system throughput. First it is shown that the system throughput is bounded by half the number of stations.

THEOREM 7.1: Let N be the number of stations in the system, then the system throughput obeys: $S \leq N/2$.

Proof: Let $T(t)$ be the set of transmission points (t',i) such that $t' \leq t$. Let $R(t)$ be the set of reception points (t',i) such that $t' \leq t$. Let (t_1,i) be a reception point in $R(t)$, then there exists a transmission point (t_2,j) which is in $T(t)$ and which corresponds to (t_1,i). This is the transmission point which corresponds to the transmission of the packet successfully received at (t_1,i). For this reason we conclude that the size of the transmission-point set is larger than or equal to the size of the reception-point set: $|T(t)| \geq |R(T)|$. In addition, the sets of transmission points and reception points must be disjoint (a station cannot transmit and receive concurrently) so the number of points in the union of $R(t)$ and $T(t)$ cannot exceed the number of points in the rectangle $N \times t$. Thus $|R(t)| + |T(t)| \leq N \cdot t$. Therefore we have:

$$\frac{|R(t)|}{t} \leq \frac{N}{2}$$

Now, since the number of packets successfully received by the time t equals to the number of points in $R(t)$, we finally have:

$$S = \lim_{t \to \infty} \frac{P(t)}{t} \leq \frac{N}{2}$$

■

The next theorem states that the system throughput is upper bounded by 2. To prove this theorem let us consider, for a moment, a system consisting of N stations connected to a *unidirectional bus*. Without loss of generality, let us assume that packets are transmitted in this system from left to right. Now let us examine the time-space domain for this unidirectional system.

LEMMA 7.2: Let d be a left diagonal in the time space domain representing the unidirectional system. Then, there exists at most one reception point on d.

Proof: For the contradiction assume that d contains more then one reception points, and let R_1 and R_2 be such two points; let R_1 be the upper point of the two (see figure 7.8).

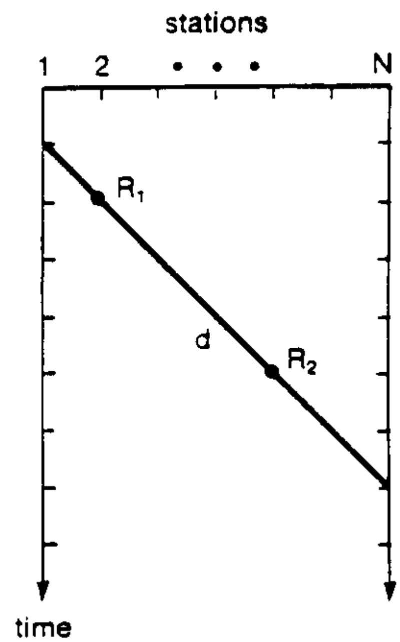

Figure 7.8: Two Reception Points in the Unidirectional System

Since all packets are transmitted left to right the transmission heard at the point R_1 must be also heard at R_2. This is in contradiction either to the assumption that each transmission is destined to exactly one station, or to the assumption that R_2 is a reception point. By the contradiction, the claim is proved. ■

LEMMA 7.3: The throughput of the unidirectional bus is $S \leq 1$.

Proof: From lemma 7.3 the number of reception points in the rectangle $t \times N$ must obey: $|R(t)| \leq t+N-1$. Thus:

$$S \leq \lim_{t\to\infty} \frac{t+N-1}{t} = 1 + \lim_{t\to\infty} \frac{N-1}{t}$$

and the proof follows. ■

Now that we examined the unidirectional bus system, let us return to the bidirectional bus system.

THEOREM 7.4: The throughput of the bidirectional bus system obeys: $S \leq 2$.

Proof: Let SYS1 be the bidirectional bus system. Let SYS2 be a system with the same number of stations, which are connected to two unidirectional busses: one is used to transmit packets from left to right and the other is used to transmit packets from right to left. Let us assume that in SYS2 each station is connected to each unidirectional bus according to the same rule a station is connected to the bidirectional bus in SYS1; i.e., a station can transmit on the bus or receive from it but cannot perform both operations concurrently. In addition, let us assume that the operations taken by a station on one bus are independent of the operations taken on the other bus; for example, a station can transmit on one bus and receive from the other bus concurrently. Clearly, the throughput of SYS2 cannot exceed the throughput of two separate unidirectional systems, so: $S(SYS2) \leq 2$. In addition, from the assumptions made above it is easy to see that the maximum achievable throughput of SYS2 is not smaller than the maximum achievable throughput of SYS1. The reason is that any operation operated on the bidirectional system can be simulated on the two unidirectional-line system. Thus, it follows that $S(SYS1) \leq 2$. ■

7.3.3 The Capacity of an Unconstrained System

In this sub-section we present a lower bound on the capacity of an unconstrained system. The lower bound presented is very close to the upper bound derived above, and thus determines the system capacity.

To prove that x is a lower bound on the system capacity one has to show that the throughput x is achievable on the system. Using the time-space domain, in figure 7.9 we depict a transmission pattern implemented on a six station system. The throughput of this pattern when implemented on the six station system yields throughput of value $S = \frac{10}{6}$. It is easy to see that this pattern can be implemented for a general N station system yielding throughput of value $S = 2 - 2/N$. From this observation we conclude:

COROLLARY 7.5: The capacity of a non constrained N station system is:

$$2 - \frac{2}{N} \leq C \leq \min(2, \frac{N}{2}) \tag{7.1}$$

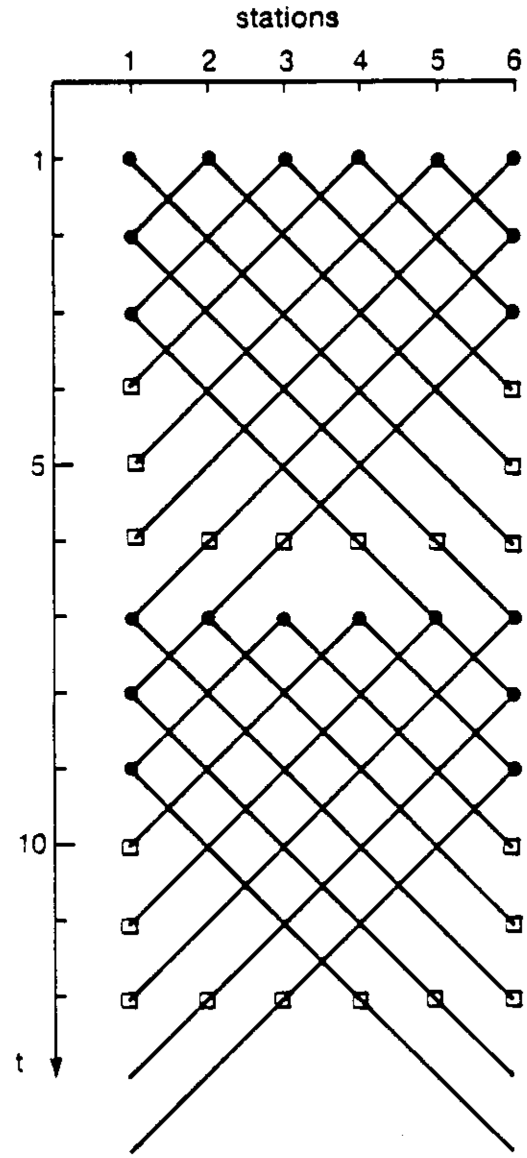

Figure 7.9: Throughput of Value 10/6 is Attainable on a Six Station System

7.3.4 The Capacity of a (Strictly) Fair System

In the previous sub-section we derived the capacity of an unconstrained system. However, the pattern used to derive a lower bound for the system capacity does not obey the restriction of either a fair system or of a strictly fair system. The reason is that according to that pattern most transmissions in the system are originated at the end stations (1 and N) and destined to these stations. Since fairness is a property which may be required from most systems it is important to see if the performance of the system is not degraded under the fairness requirement.

In figure 7.10 we depict a strictly fair transmission pattern implemented on a six station system. It is easy to check that in this pattern there is exactly one packet transmission from every station i to every other station j ($i \neq j$). Thus, if the pattern is repeatedly applied, the transmission policy is strictly fair. The throughput achieved by this particular transmission pattern is 30/22. It is easy to implement this pattern on a system consisting of an arbitrary (even *) number of stations N. The throughput attained by such a pattern can be calculated as following: First, the number of transmissions in the pattern is $N(N-1)$ (from every station to every other station, exactly one packet). Second, the time it takes to complete the pattern is the following sum:

$$\text{time} = 2+2\cdot(4+6+8+\cdots+N) = \frac{N^2+2N-4}{2} \tag{7.2}$$

Now, dividing the number of transmissions by the time we get the system throughput:

$$S = 2\cdot\frac{N^2-N}{N^2+2N-4} \tag{7.3}$$

A more efficient transmission pattern for the strictly fair system has been suggested by C. Ferguson [Ferg83]. This pattern is depicted in figure 7.11; the throughput attained by this pattern is:

$$S = 2 - \frac{4}{N+2} \tag{7.4}$$

From this result, and since the upper bound derived in sub-section 7.3.2 holds for the strictly fair system, we conclude:

COROLLARY 7.6: The capacity of a strictly fair N station system is:

$$2 - \frac{4}{N+2} \leq C \leq \min(2, \frac{N}{2}) \tag{7.5}$$

7.3.5 The Capacity of a Polite System

In this sub-section we calculate the capacity of a system where all stations are polite. As defined above a polite station is not allowed to transmit when it hears a transmission originated from another station.

The following lemma states that the number of transmission points on a diagonal (in the time-space domain) is bounded:

LEMMA 7.7: Let d be a left diagonal in the time-space domain representing a polite system.

* A similar pattern can be used for systems consisting of odd number of stations.

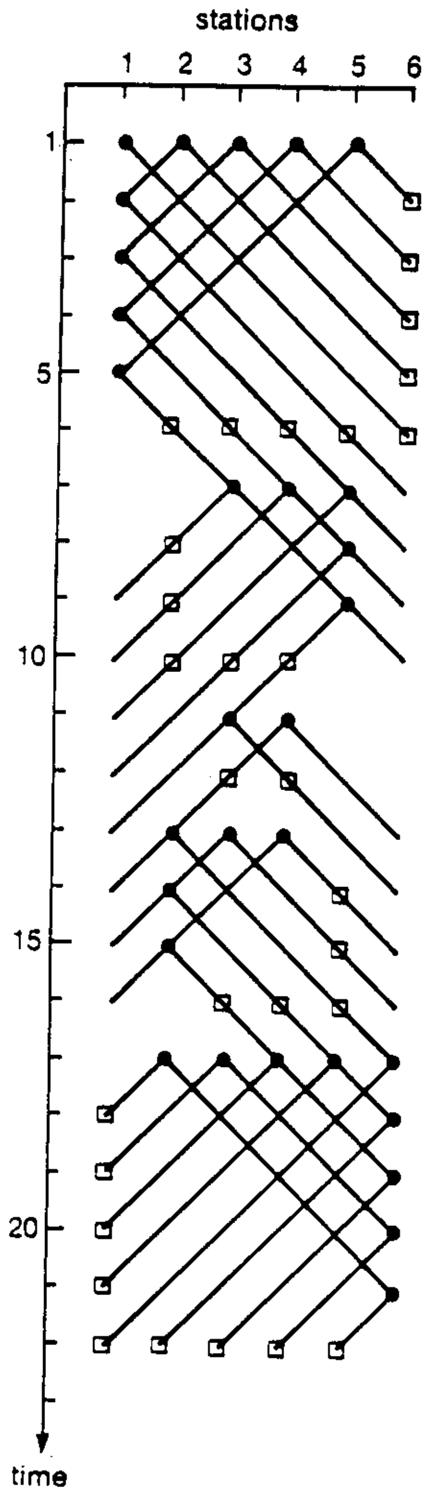

Figure 7.10: Strictly Fair Throughput of Value 30/22 is Attainable on a Six Station System

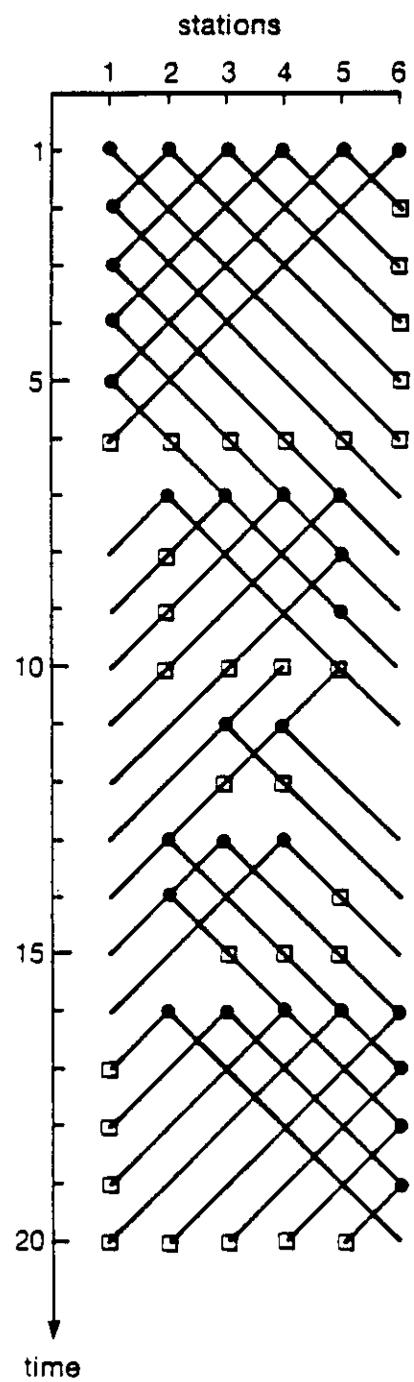

Figure 7.11: Strictly Fair Throughput of Value 30/20 is Attainable on a Six Station System

Then, there exists at most one transmission point on d.

Proof: For the contradiction assume that there exists a left diagonal with more than one transmission points on it. Let d be such a diagonal, let T_1 and T_2 be two transmission points on d, and let T_1 the upper of these points. Clearly, the transmission originated at T_1 must be heard at the point T_2, in contradiction to the assumption that the system is polite. By the contradiction the claim is proved. ■

From lemma 7.7 it is now easy to show that the capacity of a polite system is exactly 1.

THEOREM 7.8: The capacity of a polite system is exactly 1.

Proof: First we show that the system capacity is upper bounded by 1. Let $T(t)$ be the set of transmission points (t',i) such that $t' \leq t$. Let $R(t)$ be the set of reception points (t',i) such that $t' \leq t$. As shown in the proof of theorem 7.1, the size of $T(t)$ is greater or equal to the size of $R(t)$, i.e., $|T(t)| \geq R(t)$. The number of packets transmitted and successfully received by time t is $P(t)=|R(t)|$. From lemma 7.7 it is obvious that the size of $R(t)$ is bounded by the number of left diagonals in the rectangle $t \times N$, i.e., $|R(t)| \leq t+N-1$. Thus, the throughput of the system is bounded by:

$$S \leq \lim_{t \to \infty} \frac{t+N-1}{t} = 1 + \lim_{t \to \infty} \frac{N-1}{t}$$

Therefore for every $\epsilon > 0$ and for every throughput S attainable on the system, $S < 1+\epsilon$ and the capacity of the channel is upper bounded by 1.

Now, it is easy to see that throughput of value 1 is attainable in the system, this can be achieved by having station 1 transmitting all the time and all the other stations stay silent. Thus, we conclude that the capacity of the system is exactly 1. ■

After calculating the capacity of a polite system, we next discuss the capacity of a unidirectional-polite system. It is easy to see that if the direction of politeness can be chosen for every station independently of the politeness direction chosen for the other stations, then the capacity of the system can get close to 2. To verify this property observe figure 7.9: Let station 1 be polite to the left and station 6 be polite to the right (which actually implies no politeness of these stations), let station 2 be polite to the left and station 5 be polite to the right, and let stations 3 and 4 be either polite to the right or polite to the left. Under this politeness rule the transmission policy depicted in figure 7.9 is still valid and the system throughput can get as high as $2-2/N$.

On the other hand, if the politeness direction is chosen to be uniform (i.e., either all stations are polite to the left or all stations are polite to the right) then lemma 7.7 still holds* and the system capacity is 1.

7.3.6 Discussion

From the analysis made above it is evident that the potential of the fast bidirectional bus system is relatively high. The capacity of similar single shared-channel systems, like the one-hop packet radio network or the relatively-slow bidirectional bus system, is known to be 1. In comparison it was shown above that the time-space event-separation observed in the very-fast bus system allows the throughput of this system to get as high as 2. This is shown to hold even if (strict) fairness is required in the system.

However, as shown in sub-section 7.3.5, forcing on the system the politeness property, a property which increases the actual throughput of a relatively-slow bus system (like in the CSMA access scheme), decreases the system capacity down to 1. Nevertheless, applying directional politeness, does not necessarily degrades the system capacity.

The bounds on the system capacity versus the system size (number of stations in the system) is plotted in figure 7.12.

7.4 The System Performance Under Stochastic Arrivals: A No-queueing Simplistic Model

After studying the bounds on the system performance, next, in this section we study the system behavior under the assumption of stochastic arrivals.

The system model is the one given in section 7.2 above. The arrival process (i.e., the way by which packets arrive to the stations) is modeled according to the "traditional" model used in the literature (see for example, [Abra73]) of packet radio networks. According to this model, the packet transmissions of each station are modeled as a sequence of independent Bernoulli trials. This sequence represents the combined stream of old retransmitted packets and newly arriving packets. Thus we have:

$$G_i = Pr[i\text{th station transmits a packet in any given slot}] \qquad i = 1,2,\cdots,N$$

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* Lemma 7.7 holds for a a left-polite policy; clearly, a symmetric lemma holds for a right-polite policy

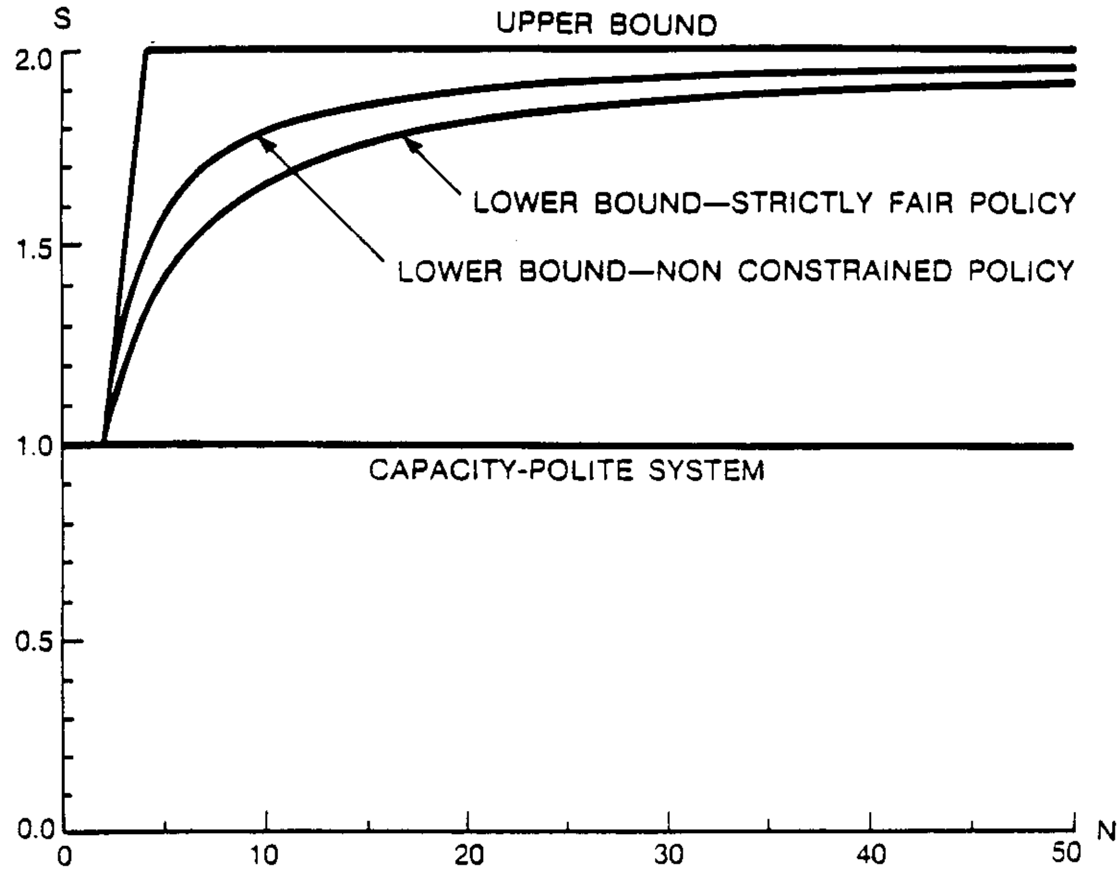

Figure 7.12: Bounds on the System Capacity

Since in our model there is importance to the packet destination*, we define further more the destination of each packet sent:

$$r_{ij} = Pr[\text{station } i\text{'s packet is destined to station } j] \qquad j \neq i$$

This definition obviously requires:

$$\sum_{j \neq i} r_{ij} = 1 \qquad i = 1,2,\cdots,N$$

* the destination information is not important in the model of one hope packet radio network, since the successful reception of a packet does not depend on its destination.

Two important parameters are considered in this model: the *average traffic* (per slot), called also the *offered load,* and the *throughput.* The offered load of station i is the expected number of packets (per slot) transmitted by this station. This is denoted above by G_i. Similarly, the offered load from station i to station j, denoted by G_{ij}, is the expected number of packets transmitted from station i to station j. From the assumption made above it is obvious that $G_{ij} = r_{ij} \cdot G_i$. The total offered load of the system, denoted by G, is the average number of packets transmitted (per slot) in the system. Obviously we have $G = \sum_{i=1}^{N} G_i$.

In a similar way we define the throughput of the system. The throughput of station i, denoted by S_i, is the expected number of packets (per slot) originated at station i and successfully received at their destination. The throughput from station i to station j, denoted by S_{ij}, is the expected number of packets (per slot) successfully transmitted from station i to station j. The total system throughput, denoted by S, is the expected number of packets (per slot) transmitted and successfully received in the system. Note that this definition of throughput is consistent with the definition given is section 7.3 above.

7.4.1 Exact Throughput Analysis of a Non Polite System

We start the throughput analysis of the system by studying the non-polite scheme. In a non-polite scheme the behavior of one station is independent of the transmissions of the other stations; thus, using the model described above it is easy to calculate the system throughput.

Let i and j be two stations in the system and let $i<j$. To derive the throughput from station i to station j let us examine a time slot t and calculate the probability that at this slot station j successfully receives a packet from station i. This event occurs *if and only if* the following conditions hold:

1. At time $t+i-j$ station i transmits a packet destined to station j. This occurs with probability $r_{ij} \cdot G_i$.

2. Station j does not transmit at time t.

3. For every station k such that $k<j$ and $k \neq i$, station k does not transmit at time $t+k-j$. The probability that station k does not transmit at that slot is $1-G_k$.

4. For every station k such that $k>j$, station k does not transmit at time $t+j-k$. The probability that station k does not transmit at that slot is $1-G_k$.

Now since all events in the system are independent of each other, the probability that station j successfully receives a packet from station i (at time t) is simply the product of the probabilities given above. Since this product is independent of t, the throughput from station i to station j is equal to this product:

$$S_{ij} = G_i \cdot r_{ij} \cdot \prod_{k \neq i}^{N} (1-G_k) \qquad i \neq j \tag{7.6}$$

From (7.6) it follows that the total throughput originated at station i is:

$$S_i = \sum_{j \neq i} S_{ij} = G_i \prod_{k \neq i} (1-G_k) \qquad i=1,2,\cdots,N \tag{7.7}$$

This is exactly the throughput of the slotted Aloha system derived by [Abra73].

When the stations assumed to be symmetric, i.e., when $G_{ij} = G_{kl} = G$ for every $i \neq j$ and $k \neq l$, then the system throughput is:

$$S = NG(1-G)^{N-1} \tag{7.8}$$

This expression is maximized when $G = 1/N$, so the maximum attainable throughput in a symmetric system is:

$$S_{\max} = \left(\frac{N-1}{N}\right)^{N-1} \tag{7.9}$$

From equation (7.8) we may conclude that the throughput in a non polite system is exactly identical to the throughput in the slotted Aloha system. For this reason we do not discuss in more detail the performance of this system; this discussion can be found in the literature dealing with the analysis of the slotted ALOHA system (see, for example, [Klei76]).

7.4.2 Polite System: An Exact Analysis of a three Station Symmetric System

After analyzing the throughput-load relationships for a non polite system we next study the stochastic behavior of a polite system. The system model is similar to the one used in sub-section 7.4.1 above. This means that queueing behavior is not represented in this model and that the transmissions of each station are modeled "like" a stream of Bernoulli trials. We say "like" since the politeness rules do not allow the transmissions of a station to be a real sequence of Bernoulli trials.

For this reason, the transmissions of each station are modeled somewhat differently from their modeling in sub-section 7.4.1. For station i it is assumed that at every slot t, in which station i is not forced to be silent by the politeness rule, i will transmit with probability G_i. Thus, if we observe the slots at which station i does not obey the politeness rule, the packets transmitted from station i behave like a stream of Bernoulli trials.

Two symmetry assumptions are used in the following analysis: 1) The transmission rate G_i for symmetric stations is assumed to be identical. Thus we assume that $G_1 = G_3 = p$ and $G_2 = q$. 2) The destination of a packet transmitted from station i is equally likely to be any of the other $N-1$ stations, i.e., $r_{ij} = \frac{1}{N-1}$ for $j \neq i$ and $r_{ii} = 0$.

Under this model an exact throughput analysis of a three station symmetric system is given next. The analysis of this system is done by constructing the Markov chain representing the system. In contrast to other studies of communication networks, in which a state represents the status of the stations (like "busy", "idle" or number of packets in the station's buffer), here the station status is not sufficient to represent the system. Rather, it is required to include the *channel status* in this representation. The reason for this is that the *current location* of a packet affects the future behavior of the system. For example, observe figure 7.3: the fact that during slot t a packet propagates from station 2 to station 3 implies that station 3 will be polite at slot $t+1$ and station 4 will be polite at slot $t+2$.

Under this requirement it is convenient to represent the system status at time t by the packets locations on the channel. In figure 7.13 we depict all the possible states in the three station system. The number of states in this system is sixteen and they are denoted by $s_a, s_b, \cdots, s_p$.

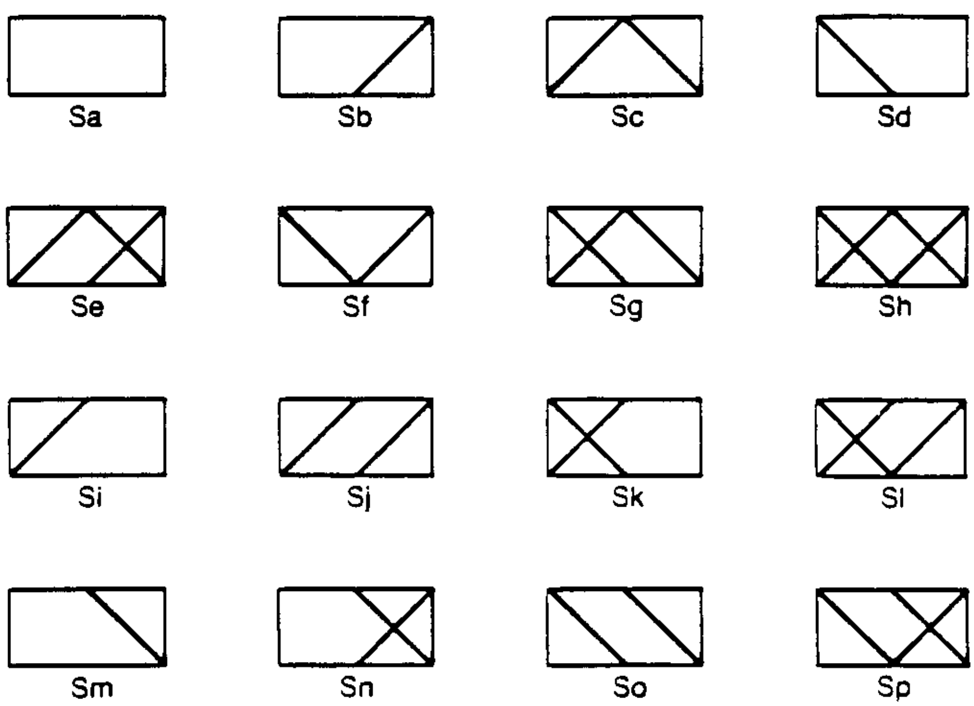

Figure 7.13: The sixteen states of the three station polite system

To better understand these states consider, for example, state s_p; this state represents concurrent propagation of three packets in the system: the first packet propagates from station 1 to station 2, the second packet propagates from station 2 to station 3 and the third packet propagates from station 3 to station 2. Note that these states represent neither the origin nor the destination of the packets. In figure 7.14 we give an example for the dynamics of the system.

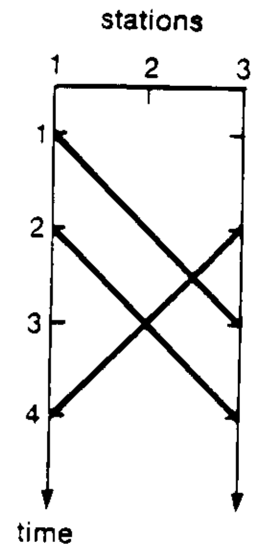

Figure 7.14: The dynamics of the system

Three packets are transmitted in this figure: station 1 transmits at slot 1 and slot 2, and station 3 transmits at slot 2. The system state is s_d at slot 1, s_p at slot 2 and s_c at slot 3.

It is easy to see that the sixteen states given above form a Markov chain. In order to find the equilibrium probabilities of the system states, we next construct the transition matrix, representing the the transitions in the system. To save work, note that some of the states in figure 7.13 are symmetric to other states. Thus, before constructing the transition matrix, we first merge the symmetric states to be represented by a single states. This is done by mapping the sixteen states $s_a, s_b, \cdots, s_p$ into ten states $s_1, s_2, \cdots, s_{10}$. This mapping is given in table 7.1.

Now, using the new merged states we construct the transition matrix P. This transition matrix is given in table 7.2. An entry in this matrix represents the probability for transition from the row state (of the entry) to the column state. The symbol $\bar{p}$ in this table stands for $1-p$ and $\bar{q}$ stands for $1-q$. When the probability for a certain transition is zero, the corresponding entry is left empty.

s_a	$\rightarrow$	s_1
s_b, s_d	$\rightarrow$	s_2
s_c	$\rightarrow$	s_3
s_e, s_g	$\rightarrow$	s_4
s_f	$\rightarrow$	s_5
s_h	$\rightarrow$	s_6
s_i, s_m	$\rightarrow$	s_7
s_j, s_o	$\rightarrow$	s_8
s_k, s_n	$\rightarrow$	s_9
s_l, s_s	$\rightarrow$	s_{10}

Table 7.1: Merging symmetric states to single states

Let $\pi_a, \pi_b, \cdots, \pi_p$ be the equilibrium probabilities for the states $s_a, s_b, \cdots, s_p$ respectively. Let $\pi_1, \pi_2, \cdots, \pi_{10}$ be the equilibrium probabilities for the states $s_1, s_2, \cdots, s_{10}$ respectively. Let π be the vector $(\pi_1, \pi_2, \cdots, \pi_{10})$ and P be the transition matrix given in table 7.2. Then the equilibrium probabilities can be found by solving the linear system:

$$\pi = \pi P \tag{7.10a}$$

$$\sum_{i=1}^{10} \pi_i = 1 \tag{7.10b}$$

After solving for π the throughput of each station can be calculated as following:

$$S_{13} = (\pi_a + \pi_b + \pi_d + \pi_f + \pi_m + \pi_n + \pi_o + \pi_p) \cdot p$$

$$= [\pi_1 + \pi_2 + \pi_5 + \cdot(\pi_7 + \pi_8 + \pi_9 + \pi_{10})] \cdot p \tag{7.11}$$

$$S_{12} = [(1-p) \cdot (\pi_a + \pi_b + \pi_d + \pi_f) + \pi_m + \pi_n + \pi_o + \pi_p] \cdot p$$

$$= [(1-p) \cdot (\pi_1 + \pi_2 + \pi_5) + (\pi_7 + \pi_8 + \pi_9 + \pi_{10})] \cdot p \tag{7.12}$$

$$S_{23} = (\pi_a + \pi_c + \pi_i + \pi_m) \cdot q$$

$$= (\pi_1 + \pi_3 + \pi_7) \cdot q \tag{7.13}$$

The three other types of throughput observed in the system can be calculated by symmetry arguments:

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	$\bar{p}^2\bar{q}$	$2\bar{p}p\bar{q}$	$\bar{p}^2q$	$2\bar{p}pq$	$p^2\bar{q}$	p^2q				
s_2							$\bar{p}^2$	$\bar{p}p$	$\bar{p}p$	p^2
s_3	$\bar{q}$		q							
s_4							1			
s_5			$\bar{p}^2$	$2\bar{p}p$		p^2				
s_6			1							
s_7	$\bar{p}\cdot\bar{q}$	$p\bar{q}$	$\bar{p}q$	pq						
s_8							$\bar{p}$	p		
s_9							$\bar{p}$		p	
s_{10}			$\bar{p}$	p						

Table 7.2: The transition matrix of the three station polite system

$$S_{31} = S_{13}\ , \qquad S_{32} = S_{12}\ , \qquad S_{21} = S_{23} \tag{7.14}$$

Using numerical methods (required to invert the transition matrix P) we have calculated from equations (7.10), (7.11), (7.12), (7.13) and (7.14) the system throughput as function of the transmission rates, p and q. The results of this analysis are given in figures 7.15, 7.16, 7.17, 7.18 and 7.19.

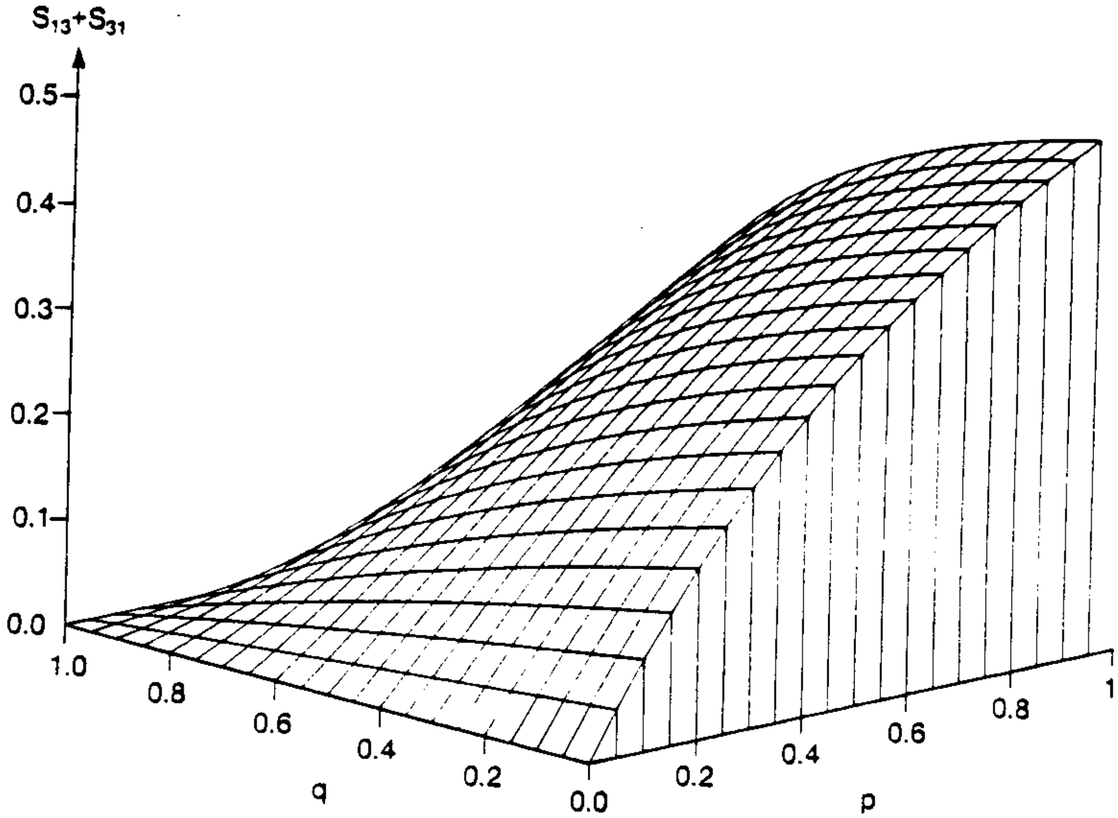

Figure 7.15: The side-to-side throughput in a three station system

Figure 7.15 is a three dimensional plot of the throughput originated at a side node and destined to the other side node (the sum of S_{13} and S_{31}) as function of p and q. To uncover the hidden lines of figure 7.15, we give in figure 7.16 a contour map corresponding to the three dimensional plot given in figure 7.15.

Figure 7.17 is a three dimensional plot of the throughput originated at a side node and destined to the middle node (the sum of S_{12} and S_{32}) as function of p and q. The shape of this plot is similar to the shape of figure 7.15, so we avoid plotting a contour map of it.

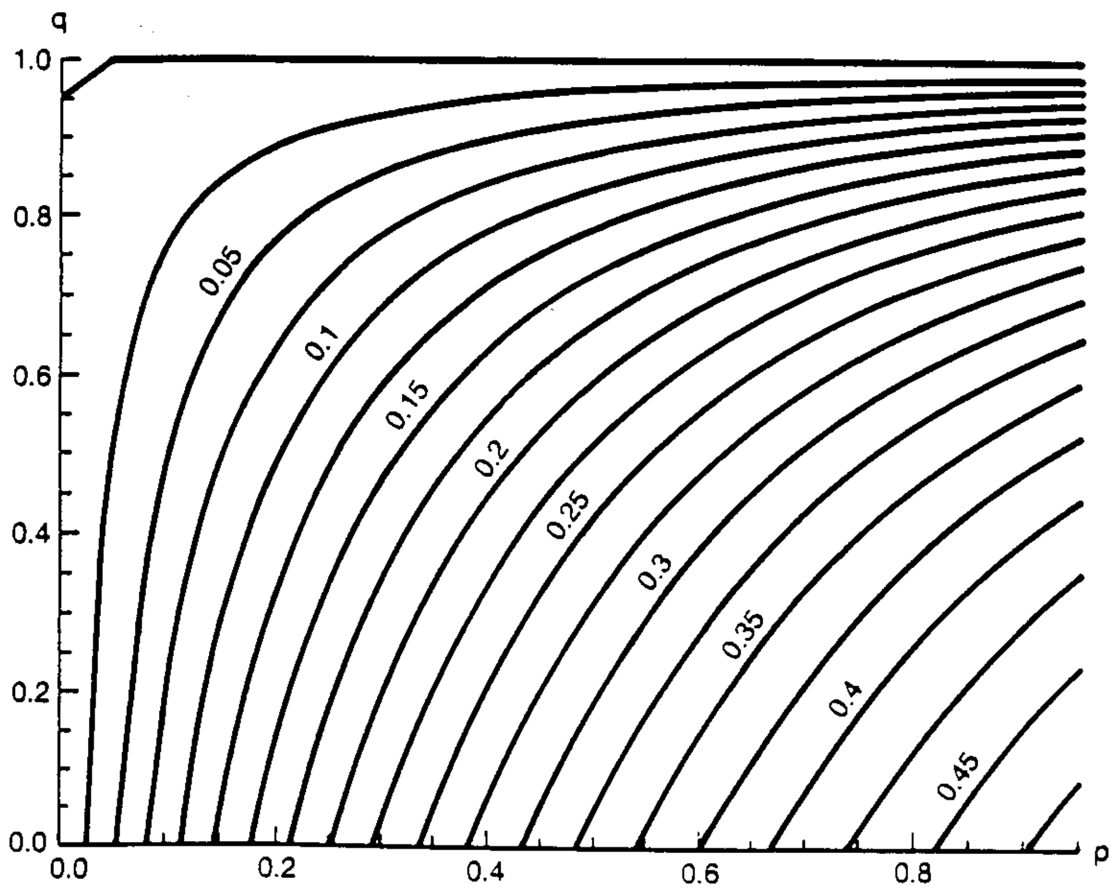

Figure 7.16: A contour map of the side-to-side throughput in a three station system

Figure 7.18 is a three dimensional plot of the throughput originated at the middle node and destined to a side node (the sum of S_{21} and S_{23}) as function of p and q.

Lastly, figure 7.19 depicts the total throughput (S) in the system as function of p and q.

A discussion of the system behavior, as observed in these figures is given in sub-section 7.4.6.

The results reported in this sub-section are important for the understanding of the system behavior. However, the drawback of the method used here is that it cannot be used to analyze the system throughput of larger systems. The reason is that the number of states in the Markov chain, according to the representation used in this sub-section, grows exponentially with

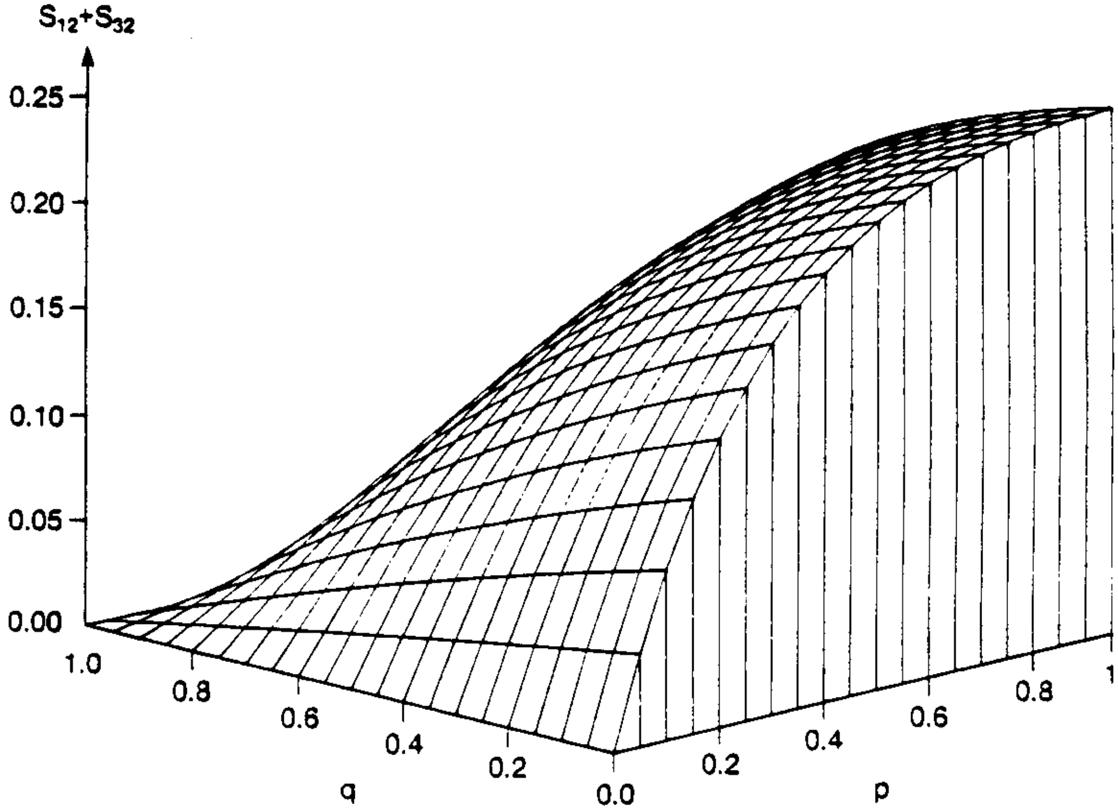

Figure 7.17: The side-to-middle throughput in a three station system

the number of stations. It is easy to see that the number of states, as function of the number of stations is: $2^{(2N-2)}$. Thus the number of states for a four station system is 64, and for a five stations system is 256 *. Since in order to solve the Markov chain one has to invert the transition matrix (an $M \times M$ matrix where M is the number of states), it is obvious that the exact method cannot be used to solve for the throughput of systems with more than five stations. For this reason in the next sub-section we suggest a method for approximating the system throughput.

*The reduction in number of states achieved by merging symmetric states is at most a factor of two, thus, even after this reduction at least $2^{(2N-3)}$ states are required.

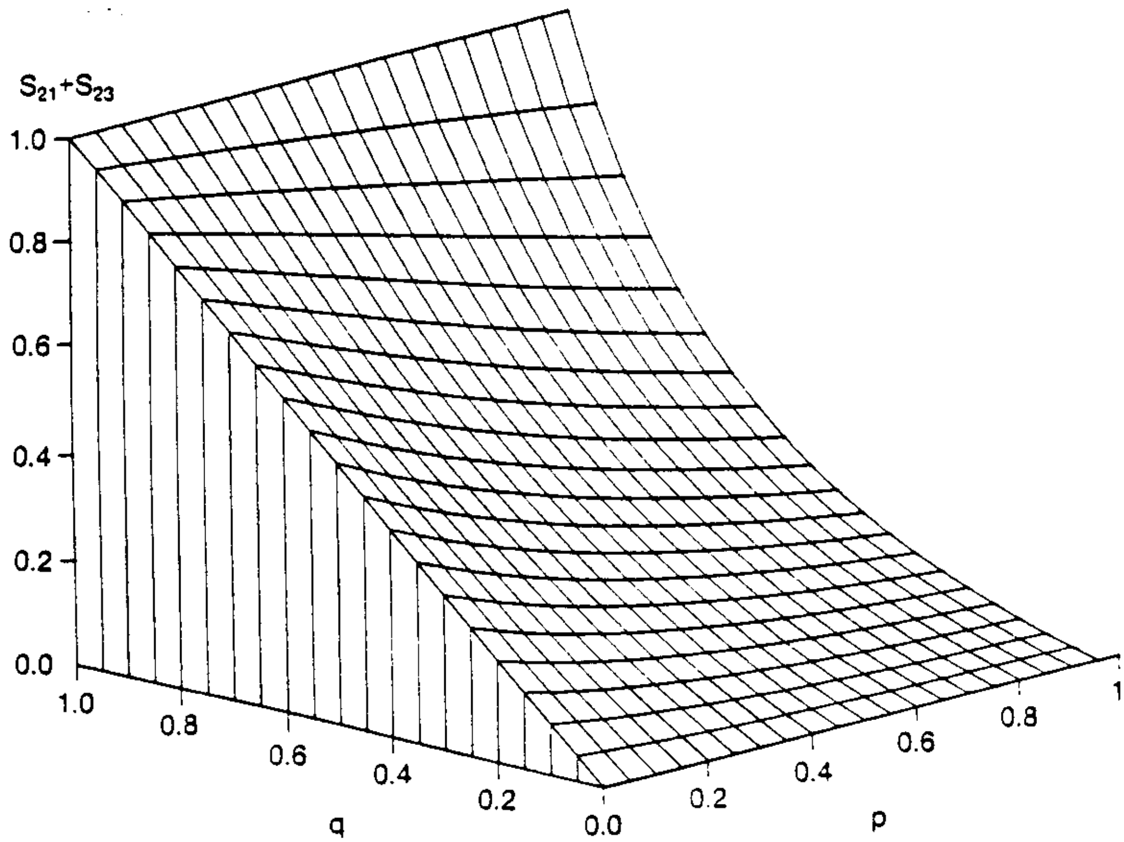

Figure 7.18: The middle-to-side throughput in a three station system

7.4.3 Polite System: An Approximation for an N station System

In this sub-section we suggest a method for approximating the throughput of a polite N station system. This method is derived from analyzing the time space domain of the N station system.

Let the triple (RS,k,t) denote the event that during slot t station k hears a packet arriving from the right. Similarly, let the triple (LS,k,t) denote the event that during slot t station k hears a packet arriving from the left. Let the triple (Q,k,t) denote the event that station k is quiet (does not transmit) at slot t.

To derive the system throughput we first calculate the probability for the event (LS,k,t) occurs. Clearly, at time t station k does not hear a packet arriving from the left *if and only if* for every station j, such that $1 \leq j < k$, station j does not transmit at time $t+j-k$. Thus, the probability of the event (LS,k,t) can be calculated as following:

$$Pr[(LS,k,t)] = Pr[(Q,k-1,t-1),(Q,k-2,t-2),\cdots,(Q,1,t-k+1)] \tag{7.15}$$

This can be calculated as:

$$Pr[(LS,k,t)] = Pr[(Q,k-1,t-1) \mid (Q,k-2,t-2),\cdots,(Q,1,t-k+1)]$$

$$\cdot Pr[(Q,k-2,t-2),\cdots,(Q,1,t-k+1)] \tag{7.16}$$

The conditional probability given above can be calculated as following:

$$Pr[(Q,k-1,t-1) \mid (Q,k-2,t-2),\cdots,(Q,1,t-k+1)]$$

$$= 1 - G_{k-1} \cdot Pr[(RS,k-1,t-1) \mid (Q,k-2,t-2),\cdots,(Q,1,t-k+1)] \tag{7.17}$$

Now to calculate the expression

$$Pr[(RS,k-1,t-1) \mid (Q,k-2,t-2),\cdots,(Q,1,t-k+1)]$$

we make the following independence assumption on the system:

The independence assumption: The event (RS,k,t) is independent of the events $(Q,k-1,t-1),\cdots,(Q,1,t-k+1)$.

This assumption means that the event that station k hears at time t a transmission arriving from the right, is independent of the fact that stations $k-1,k-2,\cdots,1$ are quiet at times $t-1,t-2,\cdots,t-k+1$, respectively. Obviously, this is not a true property of our system since these events are correlated to each other. However, it is easy to see that the dependency between these events is relatively weak and thus we assume full independence.

Let R_k denote the probability that during slot t station k does not hear a transmission arriving from the right, i.e.:

$$R_k \triangleq Pr[(RS,k,t)]$$

Similarly let:

$$L_k \triangleq Pr[(LS,k,t)]$$

Then from the independence assumption and from equations (7.16) and (7.17) we may conclude:

$$L_k = (1-R_{k-1} \cdot G_{k-1}) \cdot (1-R_{k-2} \cdot G_{k-2}) \cdots (1-R_1 \cdot G_1) \qquad ; \; k=1,2,\cdots,N-1 \tag{7.18}$$

In asymmetric way we can calculate R_k:

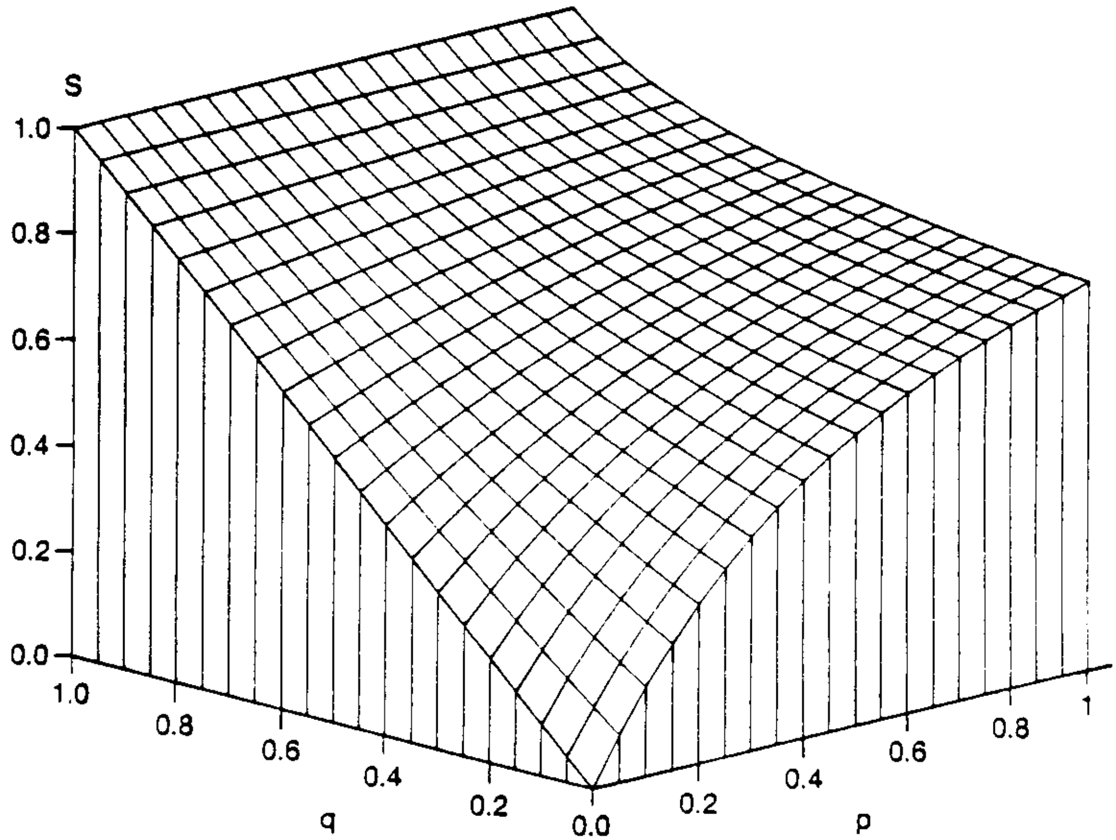

Figure 7.19: The total throughput in a three station system

$$R_k = (1-L_{k+1}\cdot G_{k+1})\cdot(1-L_{k+2}\cdot G_{k+2})\cdots(1-L_N\cdot G_N) \qquad ; \; k=2,3,\cdots,N \qquad (7.19)$$

The values of R_1 and L_N is obviously 1.

Now equations (7.18) and (7.19) form a set of $2N$-2 equations in $2N$-2 variables, a set which can be solved by numerical methods.

Having the values of $L_1,L_2,\cdots,L_k$ and of $R_1,R_2,\cdots,R_k$ we next calculate the probability that at time t station k successfully hears a packet transmitted from station j. Let us denote this event by $H(k,j,t)$ and let us assume that $j<k$. Then, $H(k,j,t)$ occurs *if and only if* the following holds: 1) At time $t-k+j$ station j does not hear a transmission arriving from the left. 2) At time $t-k+j$ station j does not hear a transmission arriving from the right. 3) At time $t-k+j$ station j transmits. 4) At time t station k does not hear a transmission arriving from the right.

Thus we have:

$$Pr[H(k,j,t)] = Pr[(LS,k,t),\sim(Q,j,t-k+j)]$$

$$= Pr[(LS,k,t), \mid \sim(Q,j,t-k+j)]\cdot Pr[\sim(Q,j,t-k+j)] \qquad (7.20)$$

From the independence assumption, it is easy to show that:

$$Pr[(LS,k,t), \mid \sim(Q,j,t-k+j)] = Pr[(LS,k,t)] = L_k \qquad (7.21)$$

Again, from the independence assumption, we have:

$$Pr[\sim(Q,j,t-k+j)] = G_j\cdot L_j\cdot R_j \qquad (7.22)$$

Thus, we finally have:

$$Pr[H(k,j,t)] = G_j\cdot L_j\cdot R_j\cdot R_k \qquad (7.23)$$

From the definition of throughput we have $S_{jk} = Pr[H(k,j,t)]$, so:

$$S_{jk} = G_j\cdot L_j\cdot R_j\cdot R_k \quad ; \qquad j<k \qquad (7.24a)$$

If $j>k$ then we have:

$$S_{jk} = G_j\cdot R_j\cdot L_j\cdot L_k \quad ; \qquad j>k \qquad (7.24b)$$

Thus from equations (7.18), (7.19), (7.24a) and (7.24b) one can calculate the system throughput as function of the transmission parameters.

If we assume that the system is symmetric, namely, that $G_1=G_2=\cdots=G_N = p$ then clearly we have:

$$R_k = L_{N-k+1} \quad ; \qquad k=1,2,\cdots,N \qquad (7.25)$$

For the symmetric system let us denote:

$$I_k \triangleq L_k$$

so we have the following set of N-1 equations:

$$I_k = (1-I_{N-k+2}\cdot p)\cdot(1-I_{N-k+3}\cdot p)\cdots(1-I_N\cdot p) \quad ; \qquad k=2,3,\cdots,N \qquad (7.26)$$

And the throughput from station j to station k is given by:

$$S_{jk} = p\cdot I_j\cdot I_{N-j+1}\cdot I_k \quad ; \qquad j\neq k \qquad (7.27)$$

7.4.5 Polite Systems: A Verification of the Approximation Method

To verify the accuracy of the approximation method suggested above, it is important to compare the results predicted by this method to the exact throughput in the system.

Testing the accuracy of the approximation method for a three station system is not difficult. The results reported in sub-section 7.4.2 give an exact calculation of the system throughput as function of the transmission parameters for a system where the side nodes are assumed to be symmetric. The approximation method yields a set of two (only two due to the symmetry of the side nodes) equations:

$$I_2 = 1 - I_3 p \tag{7.28a}$$

$$I_3 = (1 - I_3 p)\cdot(1 - I_2 q) \tag{7.28b}$$

This set can be solved analytically to yield expressions for the values of I_2 and I_3, from which the system throughput can easily be calculated:

$$S_{13} = S_{31} = I_3 p$$

$$S_{12} = S_{32} = I_3 p \cdot (1 - I_3 p)$$

$$S_{23} = S_{21} = I_2^2 q \tag{7.29}$$

To compare the approximation results to the exact results let us assume that the system is fully symmetric, i.e., $p = q$. Under this assumption we plot in figure 7.20 the system throughput as function of the transmission parameter p. Four throughput curves are depicted in this figure: 1) The throughput from a side node to the other side node ($S_{13} + S_{31}$). 2) The throughput from a side node to the middle node ($S_{12} + S_{32}$). 3) The throughput from the middle node to a side node ($S_{23} + S21$). 4) The total throughput (S). The solid line in this figure represents the exact value of the throughput and the broken line represents the approximation. From this figure we can make the following observations on the quality of the approximation:

1. The general shape of the approximation curves is very close to the shape of the exact throughput curves.

2. For the range $p < .5$ the throughput values predicted by the approximation method are a very good approximation of the true values. For the range $p > .5$ the approximation prediction is not as good.

3. The values predicted by the approximation method for the *total* throughput are very close to the true values (three percent error at most!) for every value of p.

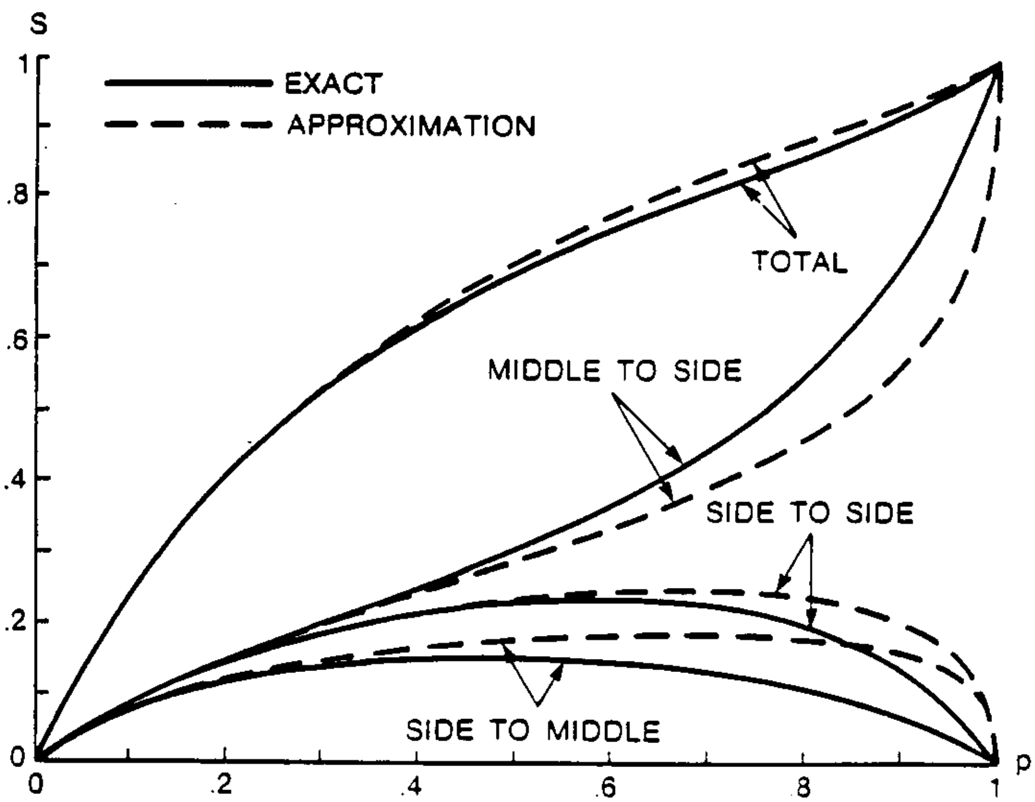

Figure 7.20: The throughput in a fully symmetric three station system

Next we examine the approximation method when applied to a five station fully symmetric system. The approximation method yields a set of four equations in four unknown variables, a set which can be solved by numerical methods. Since we cannot derive an expression for the exact throughput in the system a simulation program is used to find the true values of the system throughput. Figure 7.21 depicts the comparison between the approximation results and the simulation results for this system. Four curves of throughput are plotted in this figure: 1) The total throughput originated at station 1 (S_1). 2) The total throughput originated at station 2 (S_2). 3) The total throughput originated at station 3 (S_3). 4) The total system throughput (S). Due to symmetry arguments the other types of throughput, S_4 and S_5, are equal to S_2 and S_1, respectively. The solid lines in this figure represent the results predicted by the approximation method. The points represent the simulation results. The observations made on the approximation of the three station system, apply to this figure too. Nevertheless, it can be noted that the accuracy of the approximation for the five station system is better than that of the three station system.

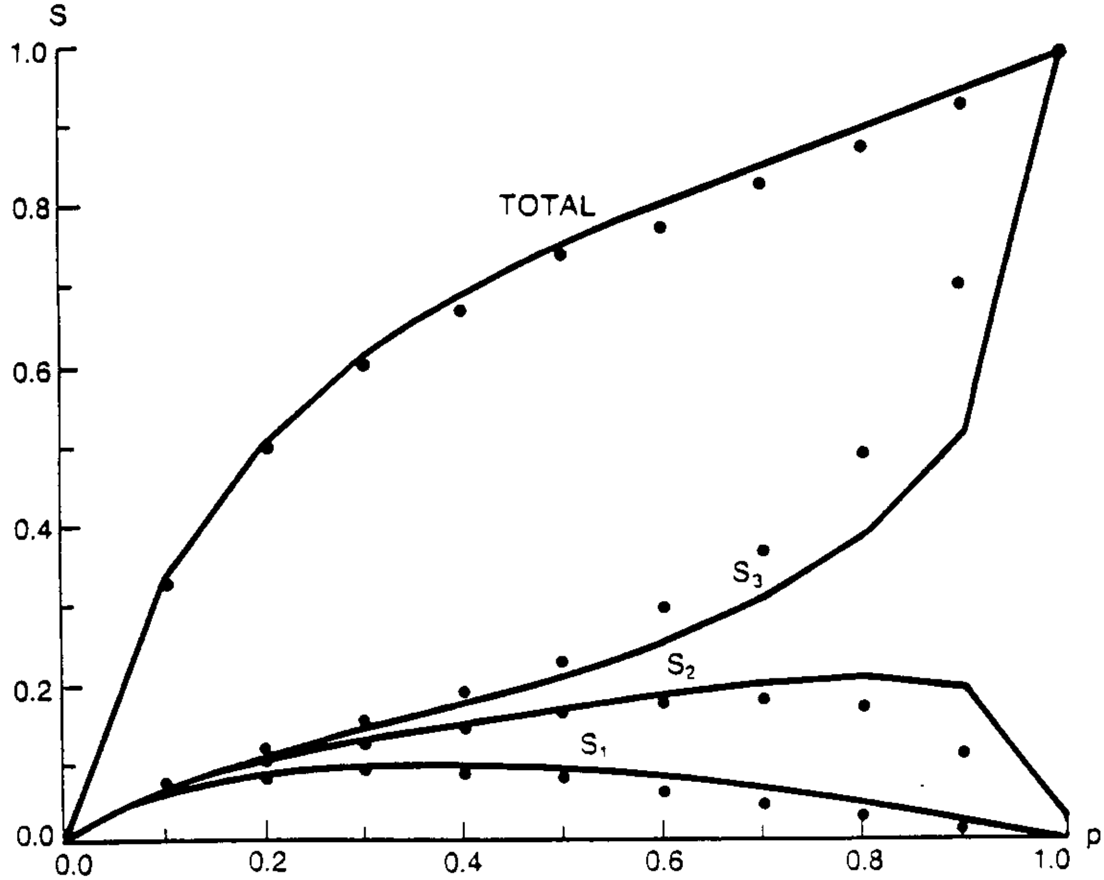

Figure 7.21: The throughput in a fully symmetric five station system

7.4.6 A Discussion of the System Performance

The analysis done in this section reveals the important properties of the very-fast bus system. These properties are discussed next.

From the throughput analysis done for the three station system we can study the system behavior under non symmetric input loads. From figures 7.15, 7.16 and 7.17 it can be seen that the throughput originated at the side stations (1 and 3) behaves as following: 1) When the offered load of the middle station (G_2) stays constant, the throughput of the side stations increases with their offered load. Note that although the two side stations compete with each

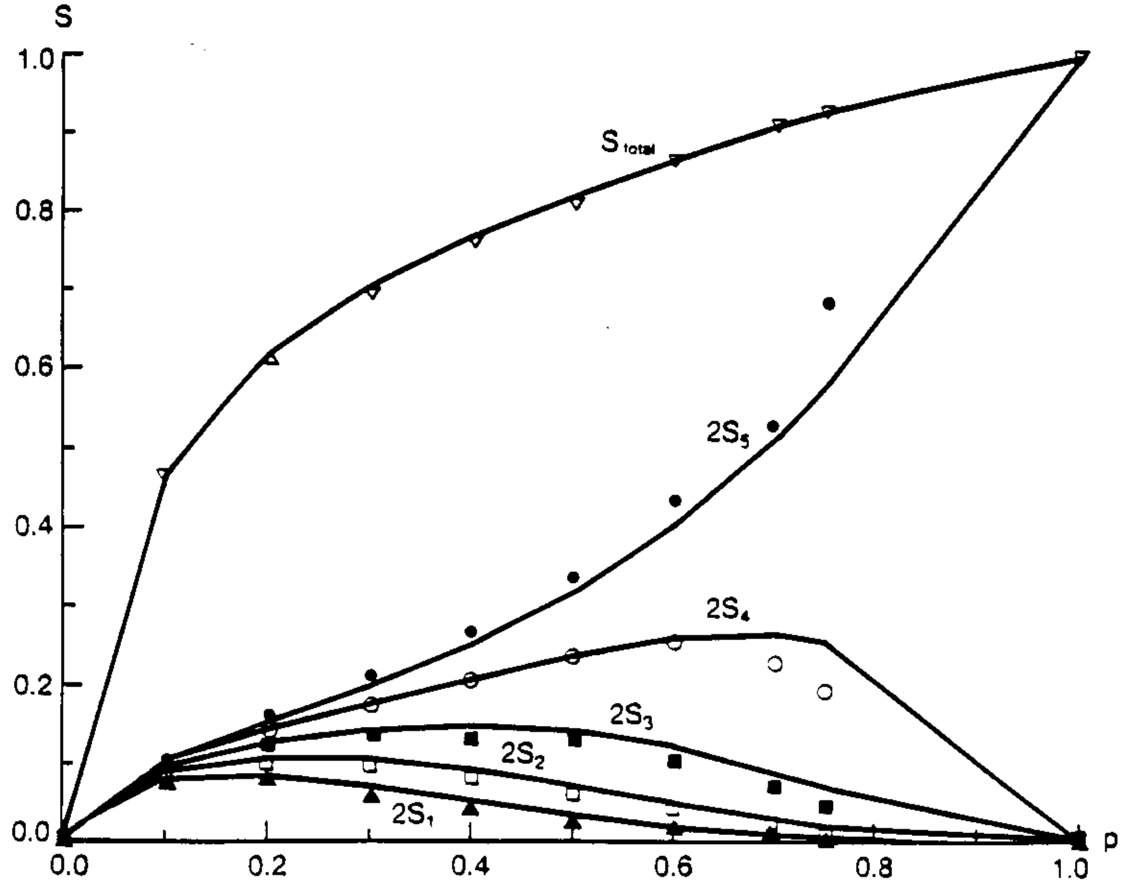

Figure 7.22: The throughput in a fully symmetric ten station system

other, their joined throughput ($S1 + S3$) increases when both stations are more active (namely, when p increases). 2) When the offered load of the side stations stays constant, the throughput of these stations decreases with the offered load of the middle station. This means that the more the middle station is active the less packets originated at the side stations will be received correctly at their destination. In a similar way, the throughput originated at the middle station behaves as following: 1) When the offered load of the side stations stays constant, the throughput of the middle station increases with the offered load of this station. 2) When the offered load of the middle station stays constant, its throughput decreases with the offered load of the side stations.

These properties can be summarized as following: When a certain station increases its load, the throughput originated at this station will increase while the throughput originated at the other stations will decrease. This behavior is quite common for shared channel communication networks; for example, the slotted ALOHA system, or the non-polite system described in sub-section 7.4.1 above behave the same way (see equation (7.7) which describes the throughput in these systems).

While at the individual station level the polite system behaves very much like other shared channel systems, the advantages of this system are revealed by examining the global behavior of the system. From figure 7.19 it is easy to see that the total throughput in the system increases both with p and with q. This means that increasing the offered load either of the side stations or of the middle stations, causes an increment in the total throughput. The importance of this property is that the system is very stable: whenever the system load increases the throughput increase too. This property is not very common in shared channel communication networks. For example, the slotted ALOHA system mentioned above is not stable (see [Klei76] for example); in that system an increase in the offered load may cause the throughput to decrease.

The importance of the stability property is that no special mechanisms are required for controlling the system stability. In the non stable systems, like the slotted ALOHA, it is required to control the offered load to prevent the system from getting into unstable situations (situations in which the system blocks itself); here these mechanisms are not required since the system controls itself in a natural way.

The explanation for this stability property can be given by observing the station behavior in the time-space domain. Let i be an arbitrary station in the system, and let us assume that at time t station i transmits a packet. Now, let us examine the behavior of this station at time $t+1$. If the offered load of station i is relatively low, the station is not likely to transmit at time $t+1$; if the offered load is relatively high, the station is likely to transmit at this slot. Thus, if the offered load is high, a successful transmission at time t will imply a sequence of successful transmissions originated at station i. This is true since stations who are polite to the packets originated from i at time t will continue being polite for all the packets transmitted from i after time t. This behavior is very similar to the behavior of the exhaustive scheme studied in chapter 5: A station who transmitted a packet is very likely to continue transmitting additional packets.

It can be observed, therefore, that at high loads the system is very likely to run itself into "self synchronization" states. In these states, a single station will successfully transmit a sequence of packets while all the other stations politely listening. In contrast, when the offered load is relatively low, the system is not likely to get synchronized, and its behavior is quite random.

When the system is fully symmetric, its behavior is very similar. Figure 7.20 shows that at low load the throughput of every station increases with the offered load. At high loads, in contrast, the throughput of the middle station increases with the offered load but the throughput of the side stations decreases with the offered load. However, the total throughput always increases with the transmission parameter p. Similar properties can be observed in the behavior of the five station system and the ten station system. This can be seen in figures 7.21 and 7.22.

Next let us compare the throughput observed in different systems. Figure 7.23 depicts the total throughput as function of the transmission parameter p, in several symmetric systems.

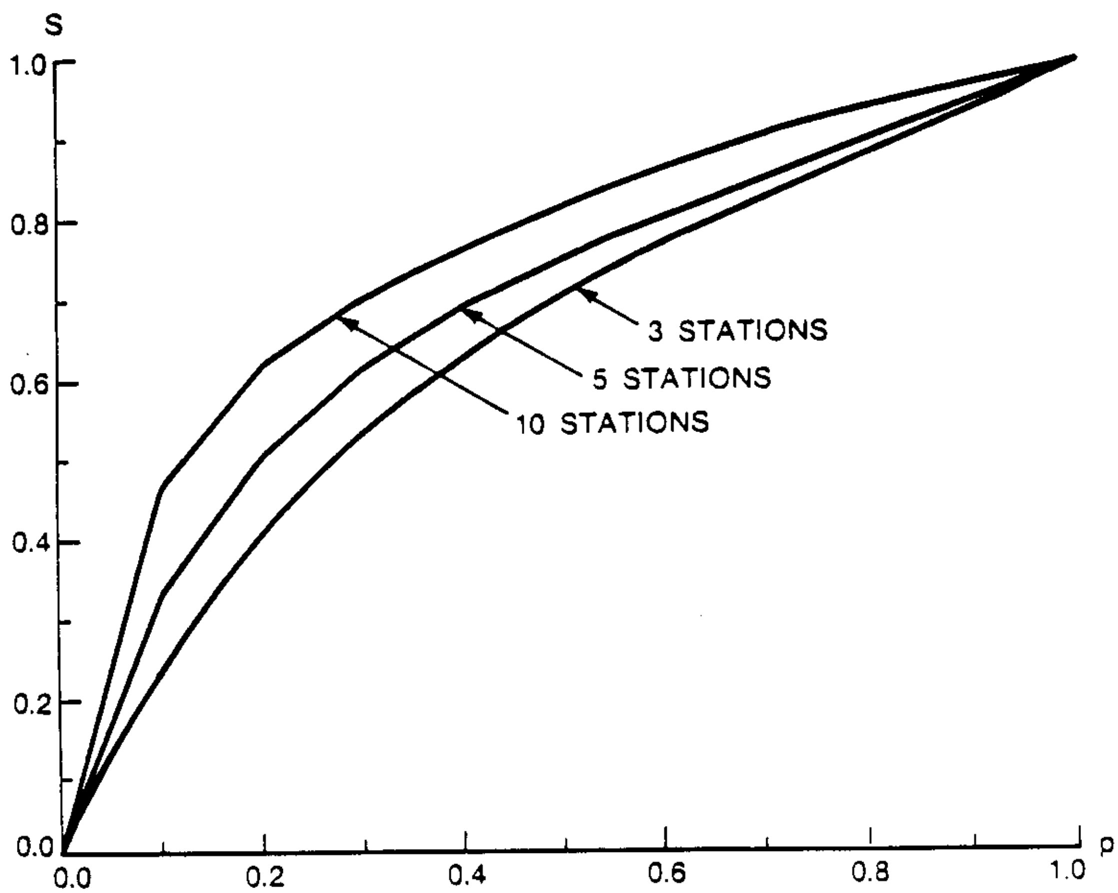

Figure 7.23: A comparison of the total throughput for different networks

The curves plotted in this figure correspond to the values calculated by the approximation method. From this figure we may learn that for a given value of the transmission parameter p, the system throughput increases with the number of stations. The reason is that by holding p constant and increasing N, we cause the system to be more "noisy" (more stations transmit with probability p) and thus force the system into synchronization.

APPENDIX A
Random Polling: The Analysis of the Exhaustive System

A.1 Gambler]s Ruin Problem

The model of the *Gambler's ruin problem* is closely related to the exhaustive service model. In the following we describe this model and review the important known results related to this problem. A detailed analysis of this problem can be found in many references, for example, see [Taka83, Konh80].

We consider a gambler who starts gambling with initial capital W_0 (≥ 0). The gambler plays a sequence of independent and identical games. The gain on the nth game is X_n and the fee for the nth game is 1 unit. The capital after the nth game is:

$$W_n = W_0 + X_1 + X_2 + \cdots + X_n - n \qquad n \geq 1$$

The *gambler's ruin time,* denoted by T, is the smallest n for which the gambler's capital becomes zero:

$$T = \min\{n : W_n = 0\}$$

The statistical measures of the initial capital and of the gains on the games are defined to be:

$$H(z) \triangleq E[z^{W_0}], \quad P(z) \triangleq E[z^{X_n}], \quad \mu \triangleq E[X_n] < 1, \quad \sigma^2 \triangleq Var[X_n]$$

An important role in the analysis of the gambler's ruin problem is played by the z-transform $\Theta(w)$. This is the z-transform of the ruin time when $W_0 = 1$, and is the solution of the following equation:

$$\Theta(w) - wP[\Theta(w)] = 0 \qquad |\Theta(w)| < 1$$

The derivatives of $\Theta(w)$ can be found from the above relation:

$$\Theta(1) = 1, \quad \Theta^{(1)}(1) = \frac{1}{1-\mu}, \quad \Theta^{(2)}(1) = \frac{\mu}{(1-\mu)^2} + \frac{\sigma^2}{(1-\mu)^3}$$

The z-transform of the ruin time can be shown to be:

$$E[w^T] = H[\Theta(w)]$$

and the first moments of the ruin time are:

$$E[T] = \frac{E[W_0]}{1-\mu}, \qquad Var[T] = \frac{Var[W_0]}{(1-\mu)^2} + \frac{\sigma^2 E[W_0]}{(1-\mu)^3}$$

An additional useful result is the following relation:

$$E\left[\sum_{n=0}^{T-1} z^{W_n}\right] = z\frac{H(z)-1}{z-P(z)}$$

A.2 Solving for f(j):

To find $f(j)$ we differentiate (4.19) with respect to z_j. This yields:

$$f(j) = p_j r_j \mu_j + \sum_{i \neq j} p_i \left[r_i \mu_j + f(j) + \frac{f(i)\mu_j}{1-\mu_i} \right] \tag{A.1}$$

Rearranging (A.1) gives:

$$f(j) = \frac{\mu_j}{p_j} \cdot \left[\sum_{i=1}^{N} p_i r_i + \sum_{i \neq j} \frac{p_i f(i)}{1-\mu_i} \right] \tag{A.2}$$

To solve (A.2) let us evaluate $f(k)$:

$$f(k) = \frac{\mu_k}{p_k} \cdot \left[\sum_{i=1}^{N} p_i r_i + \sum_{i \neq k} \frac{p_i f(i)}{1-\mu_i} \right] \tag{A.3}$$

Now, subtracting (A.3) from (A.2) gives:

$$\frac{p_j f(j)}{\mu_j} - \frac{p_k f(k)}{\mu_k} = \frac{p_k f(k)}{1\mu_k} - \frac{p_j f(j)}{1\mu_j}$$

which can be rearranged to give:

$$f(k) = f(j) \cdot \left(\frac{p_j f(j)}{\mu_j (1-\mu_j)} \right) \cdot \left(\frac{\mu_k (1-\mu_k)}{p_k} \right) \tag{A.4}$$

Now, substituting (A.4) in the right hand side of (A.2) for all values of i gives:

$$\frac{\mu_j f(j)}{p_j} = \sum_{i=1}^{N} p_i r_i + \sum_{i \neq j} \mu_i \cdot \left(\frac{p_j f(j)}{\mu_j (1-\mu_j)} \right) \tag{A.5}$$

Manipulating this expression, finally yields the value of $f(j)$:

$$f(j) = \frac{\mu_j(1-\mu_j)}{p_j} \cdot \frac{\sum_{i=1}^{N} p_i r_i}{1-\sum_{i=1}^{N} \mu_i} \tag{A.6}$$

A.3 Solving for f(i,i):

To solve for $f(i,i)$ we again note the similarity of (4.17) to the corresponding equation in the analysis of N queues exhaustively served in cyclic order (see, for example, [Taka83]). For this reason, we closely follow the approach taken in [Taka83] to calculate $f(i,i)$.

For the ease of the analysis we again condition our calculation on the queue served during the previous cycle. Differentiating (4.17) with respect to z_j and z_k gives:

$$\begin{aligned}
f(j,k \mid i) = \frac{\partial}{\partial z_j}\Big[&\Big\{\frac{\partial}{\partial z_k} R_i\Big(\prod_{n=1}^{N} P_n(z_n)\Big)\Big\}\cdot F\Big(z_1,\cdots,z_{i-1},\Theta_i\Big(\prod_{\substack{n=1\\(n\neq i)}}^{N} P_n(z_n)\Big),z_{i+1},\cdots,z_N\Big) \\
&+ R_i\Big(\prod_{n=1}^{N} P_n(z_n)\Big)\cdot\Big\{\frac{\partial}{\partial z_k} F\Big(z_1,\cdots,z_{i-1},\Theta_i\Big(\prod_{\substack{n=1\\(n\neq i)}}^{N} P_n(z_n)\Big),z_{i+1},\cdots,z_N\Big)\Big\}\Big]\Big|_{z=1} \\
= \Big[&\frac{\partial^2}{\partial z_j \partial z_k} R_i\Big(\prod_{n=1}^{N} P_n(z_n)\Big)\Big]\Big|_{z=1} \\
&+ \Big[\frac{\partial}{\partial z_k} R_i\Big(\prod_{n=1}^{N} P_n(z_n)\Big)\Big]\Big|_{z=1}\cdot\Big[\frac{\partial}{\partial z_j} F\Big(z_1,\cdots,z_{i-1},\Theta_i\Big(\prod_{\substack{n=1\\(n\neq i)}}^{N} P_n(z_n)\Big),z_{i+1},\cdots,z_N\Big)\Big]\Big|_{z=1} \\
&+ \Big[\frac{\partial}{\partial z_j} R_i\Big(\prod_{n=1}^{N} P_n(z_n)\Big)\Big]\Big|_{z=1}\cdot\Big[\frac{\partial}{\partial z_k} F\Big(z_1,\cdots,z_{i-1},\Theta_i\Big(\prod_{\substack{n=1\\(n\neq i)}}^{N} P_n(z_n)\Big),z_{i+1},\cdots,z_N\Big)\Big]\Big|_{z=1} \\
&+ \Big[\frac{\partial^2}{\partial z_j \partial z_k} F\Big(z_1,\cdots,z_{i-1},\Theta_i\Big(\prod_{\substack{n=1\\(n\neq i)}}^{N} P_n(z_n)\Big),z_{i+1},\cdots,z_N\Big)\Big]\Big|_{z=1}
\end{aligned} \tag{A.7}$$

Now, evaluating this expression, term by term, gives:

$$\Big[\frac{\partial}{\partial z_j} R_i\Big(\prod_{n=1}^{N} P_n(z_n)\Big)\Big]\Big|_{z=1} = R_i^{(1)}(1)P_j^{(1)}(1) = r_i\mu_j \qquad 1\leq j\leq N \tag{A.8}$$

$$\frac{\partial}{\partial z_i} F(z_1,\cdots,z_{i-1},\Theta_i\Big(\prod_{\substack{n=1\\(n\neq i)}}^{N} P_n(z_n)\Big),z_{i+1},\cdots,z_N) = 0 \tag{A.9}$$

$$\frac{\partial}{\partial z_j} F(z_1,\cdots,z_{i-1},\Theta_i\Big(\prod_{\substack{n=1\\(n\neq i)}}^{N} P_n(z_n)\Big),z_{i+1},\cdots,z_N)\Big|_{z=1} = \Big[\frac{\partial F_i}{\partial z_j}\Big]_{z=1} + \Big[\frac{\partial F_i}{\partial z_i}\Big]_{z=1}\cdot\Theta_i^{(1)}(1)P_j^{(1)}(1)$$

$$= f(j) + f(i)\frac{\mu_j}{1-\mu_i} \qquad j\neq i \tag{A.10}$$

$$\Big[\frac{\partial^2}{\partial z_j \partial z_k} R_i\Big(\prod_{n=1}^{N} P_n(z_n)\Big)\Big]_{z=1} = \begin{cases} \mu_j\mu_k(\delta_i^2+r_i^2) & j\neq k \\ \mu_j^2(\delta_i^2+r_i^2)+r_i(\sigma_j^2-\mu_j) & j=k \end{cases} \tag{A.11}$$

$$\Big[\frac{\partial^2}{\partial z_j \partial z_k} F(z_1,\cdots,z_{i-1},\Theta_i\Big(\prod_{\substack{n=1\\(n\neq i)}}^{N} P_n(z_n)\Big),z_{i+1},\cdots,z_N)\Big]_{z=1} \tag{A.12}$$

$$= \begin{cases} f(i)\mu_j\mu_k[\Theta_i^{(1)}(1)+\Theta_i^{(2)}(1)]+\Theta_i^{(1)}(1)[f(i,j)\mu_k+f(i,k)\mu_j] \\ \quad +f(j,k)+f(i,i)[\Theta_i^{(1)}(1)]^2\mu_j\mu_k & i\neq j,\ i\neq k,\ j\neq k \\ f(i)[\Theta_i^{(2)}(1)\mu_j^2+\Theta_i^{(1)}(1)P_j^{(2)}(1)]+2\mu_j\Theta_i^{(1)}(1)f(i,j) \\ \quad +f(j,j)+f(i,i)[\Theta_i^{(1)}(1)]^2\mu_j^2 & i\neq j=k \\ 0 & i=j \text{ or } i=k \end{cases}$$

The equations (A.8)-(A.11) can be substituted into (A.7) to yield a relation between $f(j,k\,|i)$ to the set $\{f(j,k)\}$ and the set $\{f(j)\}$. This set can be solved by numerical method to give the solution of the set $\{f(i,j)\}$.

Now, let us assume that the stations are symmetric. Under this assumption, we can now drop the subscripts from equations (A.8)-(A.11) and substitute them into (A.7). This gives:

$$f(j,k\,|i) = a+b[f(j)+f(k)]+cf(i)+d[f(i,j)+f(i,k)]+f(j,k)+d^2f(i,i)$$

$$i \neq j,\ i \neq k,\ j \neq k \tag{A.13a}$$

$$f(j,j \mid i) = a + r(\sigma^2 - \mu) + 2bf(j) + \left(\frac{\sigma^2 - \mu}{1-\mu} + c\right) f(i) + f(j,j) + 2df(i,j) + d^2 f(i,i)$$

$$i \neq j \tag{A.13b}$$

$$f(i,k \mid i) = a + b[f(k) + df(i)] \qquad i \neq k \tag{A.13c}$$

$$f(i,i \mid i) = a + r(\sigma^2 - \mu) \tag{A.13d}$$

where

$$a \triangleq \mu^2(\delta^2 + r^2)\,, \quad b \triangleq r\mu\,, \quad c \triangleq \mu^2\left[\frac{2r}{1-\mu} + \frac{1}{(1-\mu)^2} + \frac{\sigma^2}{(1-\mu)^3}\right]$$

$$d \triangleq \frac{\mu}{1-\mu}\,, \quad f^{(1)} \triangleq f(i) = \frac{N\mu r(1-\mu)}{1-N\mu} \tag{A.14}$$

Summing (A.13a) and (A.13c) over all i, we have:

$$f(j,k) = \sum_{i=1}^{N} p_i f(j,k \mid i)$$

$$= \sum_{\substack{i=1 \\ (i \neq j) \\ (i \neq k)}}^{N} p_i \{ a + b[f(j) + f(k)] + cf(i) + d[f(i,j) + f(i,k)] + f(j,k) + d^2 f(i,i)\}$$

$$+\ p_k\left\{a + b[f(j) + df(k)]\right\} +\ p_j\left\{a + b[f(k) + df(j)]\right\} \tag{A.14a}$$

Similarly, summing (A.13b) and (A.13d) over all i we get:

$$f(j,j) = \sum_{i=1}^{N} p_i f(j,j \mid i) = p_j[a + r(\sigma^2 - \mu)]$$

$$+\ \sum_{\substack{i=1 \\ (i \neq j)}}^{N} p_i\left\{ a + r(\sigma^2 - \mu) + 2bf(j) + \left(\frac{\sigma^2 - \mu}{1-\mu} + c\right) f(i) + f(j,j) + 2df(i,j) + d^2 f(i,i)\right\} \tag{A.14b}$$

Equations (A.14a) and (A.14b) form a set of N^2 linear equations, where the unknown variables are $f(i,j)$, $i=1, \cdots, N$, $j=1, \cdots, N$. Now, let us assume that the p_i's are symmetric. Due to complete symmetry in the system we have:

$$f(i) = f(j) \qquad \text{for every } i \text{ and } j$$

$$f(i,i) = f(j,j) \qquad \text{for every } i \text{ and } j$$

$$f(i,j) = f(k,l) \qquad \text{for } i \neq j \text{ and } k \neq l$$

thus, we can define:

$$f^{(1)} \triangleq f(i) \qquad i=1,\cdots N$$

$$f^{(2)} \triangleq f(i,i) \qquad i=1,\cdots N$$

$$f^{(ij)} \triangleq f(k,l) \qquad k \neq l$$

From (4.25) we know the value of $f^{(1)}$:

$$f^{(1)} = \frac{Nr\mu(1-\mu)}{1-N\mu} \tag{A.15}$$

Now, due to symmetry, $p_i = 1/N$ for every i. Using this fact and the notation defined above, (A.14a) and (A.14b) become:

$$f^{(ij)} = f(j,k) = \sum_{i=1}^{N} \frac{1}{N} f(j,k \mid i)$$

$$= \frac{N-2}{N} \cdot \left(a + 2bf^{(1)} + cf^{(1)} + 2df^{(ij)} + f^{(ij)} + d^2 f^{(2)} \right) + \frac{2}{N}\left(a + (1+d)bf^{(1)} \right) \tag{A.16a}$$

$$f^{(2)} = f(j,j) = \sum_{i=1}^{N} \frac{1}{N} f(j,j \mid i) = \frac{1}{N}\left(a + r(\sigma^2 - \mu) \right)$$

$$+ \frac{N-1}{N} \cdot \left[a + r(\sigma^2-\mu) + 2bf^{(1)} + \left(\frac{\sigma^2-\mu}{1-\mu} \right) f^{(1)} + cf^{(1)} + f^{(2)} + 2df(j,k)i + d^2 f^{(2)} \right] \tag{A.16b}$$

Rearranging (A.16a) and (A.16b) we get two equations with two unknown variables ($f^{(ij)}$ and $f^{(2)}$):

$$f^{(2)} = a + r(\sigma^2-\mu) + \frac{N-1}{N} \cdot \left[\left(2b + \frac{\sigma^2-\mu}{1-\mu} + c \right) \cdot f^{(1)} + (1+d^2) f^{(2)} + 2df^{(ij)} \right] \tag{A.17a}$$

$$f^{(ij)} = a + bf^{(1)} + \frac{N-2}{N} \cdot \left[(b+c) f^{(1)} + (1+2d) f^{(ij)} + d^2 f^{(2)} \right] + \frac{2}{N} bdf^{(1)} \tag{A.17b}$$

Solving (A.17a) and (A.17b) for $f^{(2)}$ yields:

$$f^{(2)} = \Big[r(\sigma^2-\mu)(1-\mu)\Big(N(1+2d)-N^2d\Big) \; + \; d(\mu-\sigma^2)f^{(1)}N^2$$

$$+ \; \Big((1-\mu)(2bd^2+2bd+c+2b)f^{(1)} + a(1+d)(1-\mu) + (3d+1)(\sigma^2-1)f^{(1)}\Big)\cdot N$$

$$+ \; (2bd^2+2bd+c+2b)(\mu-1)f^{(1)} + (1+2d)(1-\sigma^2)f^{(1)} \Big]\cdot\Big[\frac{1-\mu}{1-N\mu}\Big] \qquad \text{(A.18)}$$

Substituting (A.14) into (A.18) and manipulating this expression finally yields the expression for $f^{(2)}$:

$$f^{(2)} = \frac{\delta^2\mu^2N(1-\mu)}{1-N\mu} + \frac{\sigma^2rN[1-(N+1)\mu+(2N-1)\mu^2]}{(1-N\mu)^2} - \frac{Nr\mu(1-\mu)}{1-N\mu}$$

$$+ \; \frac{N^2r^2\mu^2(1-\mu)^2}{(1-N\mu)^2} + \frac{Nr^2\mu^2(N-1)(1-\mu)}{(1-N\mu)^2} \qquad \text{(A.19)}$$

APPENDIX B

Random Polling: The Analysis of the Gated System

B.2 Gated System: Solving for f(j)

To solve for $f(j)$ we use equation (4.60):

$$f(j) = \sum_{i=1}^{N} p_i\Big(r_i\mu_j + \mu_j f(i)\Big) + \sum_{\substack{i=1\\(i\neq j)}}^{N} p_i f(j) \tag{B.1}$$

Rearranging (B.1) gives:

$$\frac{p_j f(j)}{\mu_j} = \sum_{i=1}^{N} p_i\Big(r_i + f(i)\Big) \tag{B.2}$$

Thus, we have:

$$f(k) = f(j)\cdot\frac{p_j\mu_k}{\mu_j p_k} \qquad \text{for every } j \text{ and } k \tag{B.3}$$

Substituting (B.3) into the right hand side of (B.1) we get:

$$\frac{f(j)p_j}{\mu_j} = \sum_{i=1}^{N} p_i\cdot f(j)\frac{p_j\mu_i}{\mu_j p_i} + \sum_{i=1}^{N} p_i r_i \tag{B.4}$$

Solving (B.4) finally gives:

$$f(j) = \frac{\mu_j}{p_j}\cdot\frac{\sum_{i=1}^{N} p_i r_i}{1-\sum_{i=1}^{N}\mu_i} \tag{B.5}$$

B.2 Gated System: Solving for f(j,j)

To compute $f^{(2)}$ we use (4.63a), (4.63b), (4.63c) (4.63d). Assuming fully symmetric system these equations become:

$$f(j,k \mid i) = \mu^2(\delta^2+r^2)+r\mu f(j)+r\mu f(k)+f(i)\mu^2(2r+1)+f(j,k)$$

$$+\mu f(i,k)+\mu f(i,j)+\mu^2 f(i,i) \qquad i\neq j,\ i\neq k,\ j\neq k \tag{B.6a}$$

$$f(j,j \mid i) = \mu^2(\delta^2+r^2)+r(\sigma^2-\mu)+2r\mu f(j)+f(i)[\sigma^2-\mu+\mu^2(2r+1)]$$

$$+f(j,j)+2\mu f(i,j)+\mu^2 f(i,i) \qquad i\neq j \tag{B.6b}$$

$$f(j,k \mid j) = \mu^2(\delta^2+r^2)+r\mu f(k)+f(j)\mu^2(2r+1)+\mu f(j,k)+\mu^2 f(j,j)$$

$$j\neq k \tag{B.6c}$$

$$f(j,j \mid j) = \mu^2(\delta^2+r^2)+r(\sigma^2-\mu)+f(j)[\sigma^2-\mu+\mu^2(2r+1)]+\mu^2 f(j,j) \tag{B.6d}$$

Now, assuming that $p_i = 1/N$ for all i, and unconditioning (B.6) we get:

$$f^{(ij)} = f(j,k) = a_1 + a_2 f^{(ij)} + a_3 f^{(2)} \tag{B.7a}$$

$$f^{(2)} = f(j,j) = b_1 + b_2 f^{(2)} + b_3 f^{(ij)} \tag{B.7b}$$

where:

$$a_1 = \mu^2((\delta^2+r^2))+r\mu f^{(1)}+\frac{N-2}{N}\cdot r\mu f^{(1)}+f^{(1)}\mu^2(2r+1)$$

$$a_2 = (1+\mu)\cdot\frac{N-2}{N}+\mu$$

$$a_3 = \mu^2$$

$$b_1 = \mu^2((\delta^2+r^2))+r(\sigma^2-\mu)+2r\mu\frac{N-1}{N}f^{(1)}+f^{(1)}[\sigma^2-\mu+\mu^2(2r+1)]$$

$$b_2 = \mu^2+\frac{N-1}{N}$$

$$b_3 = 2\mu\frac{N-1}{N}$$

The solution of (B.7a) and (B.7b) is:

$$f^{(2)} = \frac{b_1(a_2-1)-a_1 b_3}{a_3 b_3+b_2+a_2(1-b_2)-1} \tag{B.8}$$

Evaluating the denominator of (B.8) we get:

$$a_3b_3 + b_2 + a_2(1-b_2) - 1 = \frac{2}{N^2}(1+\mu)(N\mu - 1) \tag{B.9}$$

Evaluating the numerator of (B.8) we have:

$b_1(a_2-1) - a_1b_3$

$$= \left\{\mu^2((\delta^2+r^2)) + r(\sigma^2-\mu) + 2r\mu\frac{N-1}{N}f^{(1)} + f^{(1)}[\sigma^2-\mu+\mu^2(2r+1)]\right\}\cdot\left\{(1+\mu)\frac{N-2}{N} + \mu - 1\right\}$$

$$-\left\{\mu^2((\delta^2+r^2)) + r\mu f^{(1)} + \frac{N-2}{N}\cdot r\mu f^{(1)} \cdot\cdot f^{(1)}\mu^2(2r+1)\right\}\cdot\left\{2\mu\frac{N-1}{N}\right\}$$

which, after some algebra, becomes:

$b_1(a_2-1) - a_1b_3$

$$= \frac{2}{N^2}\left\{N(f^{(1)}+r)(\sigma^2-\mu)[\mu(N-1)-1] - N\mu^2((\delta^2+r^2)) - f^{(1)}[2r\mu N(1+\mu) - 2r\mu + N\mu^2]\right\}$$

Now, substituting $f^{(1)}$ from (4.62), we get:

$$b_1(a_2-1) - a_1b_3 = \frac{2}{N^2}\cdot\left\{\frac{\sigma^2 Nr(\mu(N-1)-1)}{1-N\mu} - \frac{N^2r\mu^2(\mu+2r)}{1-N\mu} + Nr\mu + \frac{Nr\mu^2}{1-N\mu} + N\mu^2((\delta^2+r^2))\right\} \tag{B.10}$$

Now, dividing (B.10) by (B.9) we finally get:

$$f^{(2)} = \frac{\sigma^2 rN[1-(N-1)\mu]}{(1+\mu)(1-N\mu)^2} + \frac{(\delta^2-r^2)N\mu^2}{(1+\mu)(1-N\mu)}$$

$$+ \frac{(\mu+2r)N^2r\mu^2}{(1+\mu)(1-N\mu)^2} - \frac{\mu Nr}{(1+\mu)(1-N\mu)} - \frac{\mu^2 Nr}{(1+\mu)(1-N\mu)^2} \tag{B.11}$$

APPENDIX C
Random Polling: The Analysis of the Non-Exhaustive System

To solve the non-exhaustive system we need the first two derivatives of $P(z)$, $P(z)^N$, $R\left(P(z)\right)$, and $R\left(P(z)^N\right)$. These are:

$$\frac{\partial P(z)}{\partial z}\bigg|_{z=1} = \mu \ , \qquad \frac{\partial^2 P(z)}{\partial z^2}\bigg|_{z=1} = \sigma^2 - \mu + \mu^2 \tag{C.1a}$$

$$\frac{\partial \{P(z)^N\}}{\partial z}\bigg|_{z=1} = N\mu \ , \qquad \frac{\partial^2 \{P(z)^N\}}{\partial z^2}\bigg|_{z=1} = N^2\mu^2 + N\sigma^2 - N\mu \tag{C.1b}$$

$$\frac{\partial R\{P(z)\}}{\partial z}\bigg|_{z=1} = r\mu \ , \qquad \frac{\partial^2 R\{P(z)\}}{\partial z^2}\bigg|_{z=1} = (\delta^2 + r^2)\mu^2 + r(\sigma^2 - \mu) \tag{C.1c}$$

$$\frac{\partial R\{P(z)^N\}}{\partial z}\bigg|_{z=1} = Nr\mu \ , \qquad \frac{\partial^2 R\{P(z)^N\}}{\partial z^2}\bigg|_{z=1} = (\delta^2 + r^2)N^2\mu^2 + Nr(\sigma^2 - \mu) \tag{C.1d}$$

C.1 Solving for F(0,1,1,...,1):

To solve for f_0 we evaluate (4.77) at $z=1$. Using L'hospital's rule, and differentiating both the numerator and the denominator of (4.77) we get:

$$1 = F(1,1,\cdots,1) = F(0,1,1,\cdots,1)\cdot\frac{1-N\mu}{1-N\mu-Nr\mu} \tag{C.2}$$

from which (4.80) immediately follows.

C.2 The Derivation of Equation (4.83):

Differentiating (4.77) with respect to z at $z=1$ we get:

$$\frac{\partial F(z,z,\cdots,z)}{\partial z} = (N-1)f_1\cdot\left[\frac{R\left(\{P(z)\}^N\right)\cdot\left(z-\{P(z)\}^N\right)\cdot}{z-R\left(\{P(z)\}^N\right)\cdot\{P(z)\}^N}\right] + f_0\cdot\frac{\partial}{\partial z}\left[\frac{R\left(\{P(z)\}^N\right)\cdot\left(z-\{P(z)\}^N\right)\cdot}{z-R\left(\{P(z)\}^N\right)\cdot\{P(z)\}^N}\right] \quad \text{(C.3)}$$

To evaluate (C.3) we need the following derivatives:

$$\frac{\partial}{\partial z}\left[R\left(\{P(z)\}^N\right)\cdot\left(z-\{P(z)\}^N\right)\cdot\right]_{z=1} = 1-N\mu \quad \text{(C.4a)}$$

$$\frac{\partial^2}{\partial z^2}\left[R\left(\{P(z)\}^N\right)\cdot\left(z-\{P(z)\}^N\right)\cdot\right]_{z=1} = 2(1-N\mu)Nr\mu-[N^2\mu^2+N(\sigma^2-\mu)] \quad \text{(C.4b)}$$

$$\frac{\partial}{\partial z}\left[z-R\left(\{P(z)\}^N\right)\cdot\{P(z)\}^N\right]_{z=1} = 1-Nr\mu-N\mu \quad \text{(C.4c)}$$

$$\frac{\partial^2}{\partial z^2}\left[z-R\left(\{P(z)\}^N\right)\cdot\{P(z)\}^N\right]_{z=1} = -[(\delta^2+r^2)(N^2\mu^2)+Nr(\sigma^2-\mu)+2rN^2\mu^2+N^2\mu^2+N(\sigma^2-\mu)] \quad \text{(C.4d)}$$

Applying L'hospital rule to (C.3) we get:

$$\frac{\partial F(z,z,\cdots,z)}{\partial z}\bigg|_{z=1} = (N-1)f_1\cdot\left[\frac{1-N\mu}{1-Nr\mu-N\mu}\right]$$

$$+ f_0\cdot\left[\frac{\{2(1-N\mu)Nr\mu-[N^2\mu^2+N(\sigma^2-\mu)]\}\cdot\{1-Nr\mu-N\mu\}}{2(1-Nr\mu-N\mu)^2}\right]$$

$$+ f_0\cdot\left[\frac{\{(\delta^2+r^2)(N^2\mu^2)+Nr(\sigma^2-\mu)+2rN^2\mu^2+N^2\mu^2+N(\sigma^2-\mu)\}\cdot\{1-N\mu\}}{2(1-Nr\mu-N\mu)^2}\right] \quad \text{(C.5)}$$

Manipulating this expression and substituting f_0 from (4.80) we then get (4.83):

$$N\cdot\frac{\partial F(z,1,1,\cdots,1)}{z}\bigg|_{z=1} = \frac{(N-1)(1-N\mu)f_1}{1-N\mu-Nr\mu}+\frac{Nr\sigma^2}{2(1-N\mu)(1-N\mu-Nr\mu)}+\frac{N^2\mu^2\delta^2}{2(1-N\mu-Nr\mu)}+\frac{Nr\mu}{2}$$

C.3 The Derivation of Equation (4.84):

Recalling Equation (4.79):

$$F(z,1,1,\cdots,1) = \frac{(N-1)zR\left(P(z)\right)\cdot\left(1-P(z)\right)F(z,0,1,1,\cdots,1)}{Nz-R\left(P(z)\right)\cdot P(z)\cdot\left(1+(N-1)z\right)} + \frac{R\left(P(z)\right)\cdot\left(z-P(z)\right)\cdot F(0,1,1,\cdots,1)}{Nz-R\left(P(z)\right)\cdot P(z)\cdot\left(1+(N-1)z\right)}$$

To evaluate the derivative of (4.79) with respect to z at $z=1$ we need the following derivatives:

$$\frac{\partial}{\partial z}\left[R\left(P(z)\right)\cdot\left(z-P(z)\right)\right]_{z=1} = 1-\mu \tag{C.6a}$$

$$\frac{\partial^2}{\partial z^2}\left[R\left(P(z)\right)\cdot\left(z-P(z)\right)\right]_{z=1} = 2r\mu(1-\mu)-(\sigma^2-\mu+\mu^2) \tag{C.6b}$$

$$\frac{\partial}{\partial z}\left[Nz-R\left(P(z)\right)\cdot P(z)\cdot\left(1+(N-1)z\right)\right]_{z=1} = 1-N\mu-Nr\mu \tag{C.6c}$$

$$\frac{\partial^2}{\partial z^2}\left[Nz-R\left(P(z)\right)\cdot P(z)\cdot\left(1+(N-1)z\right)\right]_{z=1}$$

$$= -\left[N\{(\delta^2+r^2)\mu^2+r(\sigma^2-\mu)\}+2Nr\mu^2+2r\mu(N-1)+N(\sigma^2-\mu+\mu^2)+2\mu(N-1)\right] \tag{C.6d}$$

$$\frac{\partial}{\partial z}\left[zR\left(P(z)\right)\cdot\left(1-P(z)\right)\right]_{z=1} = -\mu \tag{C.6e}$$

$$\frac{\partial^2}{\partial z^2}\left[zR\left(P(z)\right)\cdot\left(1-P(z)\right)\right]_{z=1} = -\left[2\mu+2r\mu^2+(\sigma^2-\mu+\mu^2)\right] \tag{C.6f}$$

Now we differentiate (4.79) with respect to z and evaluate the derivative at $z=1$. Using (C.6) we get the derivative of the second term in the right hand side of (4.79):

$$\frac{\partial}{\partial z}\left[\frac{R\left(P(z)\right)\cdot\left(z-P(z)\right)\cdot F(0,1,1,\cdots,1)}{Nz-R\left(P(z)\right)\cdot P(z)\cdot\left(1+(N-1)z\right)}\right]_{z=1} = f_0\cdot\frac{\left(2r\mu(1-\mu)-(\sigma^2-\mu+\mu^2)\right)\cdot\left(1-N\mu-Nr\mu\right)}{2\left(1-N\mu-Nr\mu\right)^2}$$

$$+\ f_0\cdot\frac{\left(1-\mu\right)\cdot\left[N\{(\delta^2+r^2)\mu^2+r(\sigma^2-\mu)\}+2Nr\mu^2+2r\mu(N-1)+N(\sigma^2-\mu+\mu^2)+2\mu(N-1)\right]}{2\left(1-N\mu-Nr\mu\right)^2}$$

which, after manipulation, becomes:

$$= \frac{Nr\mu^3+(N-1)(\sigma^2+\mu-\mu^2)+Nr\sigma^2+Nr\mu-2Nr\mu^2+N\mu^2(\delta^2-r^2)\cdot(1-\mu)}{2(1-N\mu)(1-N\mu-Nr\mu)} \tag{C.7}$$

Differentiating the first term in the right hand side of (4.79) we have:

$$\frac{\partial}{\partial z}\left[\frac{(N-1)zR\left(P(z)\right)\cdot\left(1-P(z)\right)F(z,0,1,1,\cdots,1)}{Nz-R\left(P(z)\right)\cdot P(z)\cdot\left(1+(N-1)z\right)}\right]_{z=1}$$

$$= (N-1)\cdot\frac{(-\mu)\cdot f_1}{1-N\mu-Nr\mu}\ +\ (N-1)f_0\cdot\frac{-\left[2\mu+2r\mu^2+(\sigma^2-\mu+\mu^2)\right]\cdot\left[1-N\mu-Nr\mu\right]}{2(1-N\mu-Nr\mu)^2}$$

$$-\ (N-1)f_0\cdot\frac{-\left[N\{(\delta^2+r^2)\mu^2+r(\sigma^2-\mu)\}+2Nr\mu^2+2r\mu(N-1)+N(\sigma^2-\mu+\mu^2)+2\mu(N-1)\right]\cdot(-\mu)}{2(1-N\mu-Nr\mu)^2}$$

which after manipulation becomes:

$$= (N-1)\cdot\left[\frac{(-\mu)\cdot f_1}{1-N\mu-Nr\mu} - \frac{f_0[\sigma^2+\mu-\mu^2-Nr\mu^3+N\mu^3(\delta^2-r^2)]}{2(1-N\mu-Nr\mu)^2}\right] \tag{C.8}$$

Now, summing (C.7) and (C.8) we get from (4.79):

$$\left.\frac{\partial F(z,1,1,\cdots,1)}{\partial z}\right|_{z=1} = \frac{-(N-1)\mu f_1}{1-N\mu-Nr\mu} + \frac{N^2r\mu^3+N\mu^2(1-N\mu)(\delta^2-r^2)-2Nr\mu^2+(\sigma^2+\mu)Nr}{2(1-N\mu)(1-N\mu-Nr\mu)} \tag{C.9}$$

which is equivalent to equation (4.84).

C.4 The Derivation of Equation (4.85):

To solve for f_1 we use (4.81) to equate the right hand side of (C.5) with the right hand side of (C.9). This yields:

$$\frac{(N-1)f_1}{1-N\mu-Nr\mu} = \frac{N\cdot\left(N^2r\mu^3+N\mu^2(1-N\mu)(\delta^2-r^2)-2Nr\mu^2+(\sigma^2+\mu)Nr\right)}{2(1-N\mu)(1-N\mu-Nr\mu)} + \frac{-[Nr\sigma^2+N^2\mu^2\delta^2(1-N\mu)+Nr\mu(1-N\mu)(1-N\mu-Nr\mu)]}{2(1-N\mu)(1-N\mu-Nr\mu)} \tag{C.10}$$

Manipulation of this equation gives the value of f_1:

$$f_1 = \frac{(\sigma^2+\mu)Nr}{2(1-N\mu)}$$

C.5 The Derivation of Equation (4.89):

We recall that customers arrive to the system in bulks and that $X_i(t)$ denotes the size of the bulk arriving to queue i at time t. We also recall that under the symmetry assumption we have:

$$E[X_i(t)] = \mu\ , \qquad Var[X_i(t)] = \sigma^2\ ; \qquad\qquad i=1,2,\cdots,N$$

Now we look at a tagged customer, let say C_j, who arrives to queue i at time t. We denote by V_i the number of customers arriving to queue i at time t but queued behind C_j. similarly, V_i denotes the number of customers arriving to queue i at time t but queued in front of C_j. It is clear that V_i and V_i have the same statistical characteristic (symmetry) thus,

$$E[V_i] = E[V_i]$$

For this reason we can calculate $E[V_i]$. To calculate $E[V_i]$, we condition on the size of the bulk in which C_j arrives:

$$E[V_i \mid X_i(t)=k] = \frac{1}{k}\cdot\sum_{l=1}^{k} l\text{-}1 = \frac{k-1}{2} \tag{C.11}$$

Now, unconditioning (C.11) we get:

$$E[V_i] = \sum_{k=1}^{\infty}\frac{k-1}{2}\cdot\frac{k\cdot Pr[X_i(t)=k]}{\sum_{l=1}^{\infty} l\cdot Pr[X_i(t)=l]}$$

$$= \frac{1}{2\mu}\cdot\left[\sum_{k=1}^{\infty} k^2\cdot Pr[X_i(t)=k] - \sum_{k=1}^{\infty} k\cdot Pr[X_i(t)=k]\right]$$

$$= \frac{\sigma^2+\mu^2-\mu}{2\mu} \tag{C.12}$$

which is identical to (4.89).

APPENDIX D

Glossary of Notation for Chapter 4

C_i — the length of a cycle (system in equilibrium). Sometimes, when the context allows, C_i is also used to denote the ith customer.

$C_i(z)$ — the z-transform of the length of a cycle (system in equilibrium).

$F(z_1,z_2, \cdots ,z_N)$ — the z-transform of the number of customers found in the system at polling instants.

$F_i(z)$ — the z-transform of the number of customers found at queue i at polling instants.

G_i — the expected number of customers found at queue i when service completed at station i (applicable only to non-exhaustive systems).

I_i — the length of an idle period (system in equilibrium).

$I_i(z)$ — the z-transform of the length of an idle period (system in equilibrium).

L_i^* — number of customers found at queue i at polling instants (system in equilibrium).

$L_i(t)$ — number of customers at queue i at time t.

p_i — the probability that station i is polled at a given polling instant.

$P_i(z)$ — z-transform of $X_i(t)$.

$Q_i(z)$ — the z-transform of the number of customers found at queue i at arbitrary moments (system in equilibrium).

r_i — the expected length of the switch-over period associated with station i.

$R_i(z)$ — z-transform of the length of the switch-over period associated with station i.

s_0 — the probability that queue i is empty at switch-over instants (symmetric system).

S_i — the length of a service period (system in equilibrium).

$S_i(z)$ — the z-transform of the length of a service period (system in equilibrium).

T_i — the waiting time of an arbitrary customer arriving to station i (system in equilibrium).

$T_i(z)$ — the z-transform of T_i.

V_i — number of customers which arrive together (in the same bulk) with a tagged customer to queue i and which are served in front of the tagged customer.

$V_i(z)$ — the z-transform of V_i.

V_i — number of customers which arrive together (in the same bulk) with a tagged customer to queue i and which are served after the tagged customer.

W_i	the waiting time of the first customer of an arbitrary bulk arriving to station i (system in equilibrium).
$W_i(z)$	the z-transform of W_i.
z_0	the probability that no customer arrives at queue i at time t (symmetric system).
$X_i(t)$	number of arrivals to queue i at time t.
δ_i^2	the variance of the length of the switch-over period associated with station i.
μ_i	$E[X_i(t)]$
σ_i^2	$Var[X_i(t)]$
$\underline{\tau}(m)$	the instant at which the mth service period of the system iu starts.
$\tau(m)$	the instant at which the mth service period of the system terminates.
$\overline{\tau}(m)$	the instant at which the mth switch-over period of the system terminates.
$\underline{\tau}_i(m)$	the instant at which the mth service period of queue iu starts.
$\tau_i(m)$	the instant at which the mth service period of queue i terminates.
$\overline{\tau}_i(m)$	the instant at which the mth switch-over period of queue i terminates.

References

[Abra70] N. Abramson, "The ALOHA System — Another Alternative for Computer Communications," *AFIPS Conference Proceedings, FJCC* **37**, pp.281-285 (1970).

[Abra73] N. Abramson, "Packet Switching with Satellites," *AFIPS Conference Proceedings, NCC* **42**, pp.695-703 (June 1973).

[Avi-65] B. Avi-Itzhak, W. L. Maxwell, and L. W. Miller, "Queueing with Alternating Priorities," *Operations Reserch* **13**(2), pp.306-318 (April-May 1965).

[Boor80] R. R. Boorstyn and A. Kershenbaum, "Throughput Analysis of Multihop Packet Radio," *Conference Record, International Conference on Communications* (June 1980).

[Cohe83] J.W. Cohen and O.J. Boxma, *Boundary Value Problems in Queueing System Analysis* , North-Holland , Amsterdam (1983).

[Coop70] R. B. Cooper, "Queues Served in Cyclic Order: Waiting Times," *Bell System Technical Journal* **49**, pp.399-413 (1970).

[Dosh83] B. T. Doshi, *Stochastic Decomposiotion in a GI/G/1 Queue with Vacations,* Bell Laboratories, Holmdel N.J. 07730 (1983).

[Ferg83] C. Ferguson, *Private Communication,* UCLA, Los Angeles, CA (August 1983).

[Frat81] L. Fratta, F. Borgonovo, and F.A. Tobagi, "The EXPRESS-NET: A Local Area Coomunication Network Integrating Voice and Data," *Proceeding International Conference on Performance of Data Communication Systems and Their Applications* (September 1981).

[Fuhr83] S. W. Fuhrmann, *A Note on the M/G/1 Queue with Server Vacations* , Bell Laboratories, Holmdel N.J. 07730 (1983). Accepted for publication in Operations Research.

[Gall79] R. G. Gallager, P. A. Humblet, and P. M. Spira, "A Distributed Algorithm for Minimum Weight Spanning Trees," LIDS-P-906-A, MIT Laboratory for Information and Decision Systems, Cambridge, Massachusetts (October 1979).

[Gerl83a] M. Gerla, C. Yeh, and P. Rodrigues, "A Token Protocol for High Speed Fiber Optics Local Networks," in *Proceedings Optical Fiber Communication Conference*, New Orleans, Louisiana (February 1983).

[Gerl83b] M. Gerla, P. Rodrigues, and C. Yeh, "Buzz-net: A Hybrid Random Access/Virtual Token Local Network," in *Proceedings Globecom '83*, San-Diego, CA (December 1983).

[Heym77] D. P. Heyman, "The T-Policy for the M/G/1 Queue," *Management Science* **23**, pp.775-778 (1977).

[Hofr84] M. Hofri and A. Konheim, "The Analysis of Finite Quasi-symmetric ALOHA Network with Reservation," pp. 505-525 in *Proceedings IFIP*, Zurich (1984).

[Klei64] L. Kleinrock, *Communication Nets, Stochastic Message Flow and Delay*, McGraw-Hill, New York (1964). Out of print. Reprinted by Dover Publications, 1972.

[Klei74] L. Kleinrock and F. A. Tobagi, "Carrier Sense Multiple Access for Packet Switched Radio Channels," *Conference Record, International Conference on Communications*, pp.21B-1 to 21B-7 (June 1974).

[Klei75a] L. Kleinrock, *Queueing Systems, Vol I., Theory*, Wiley-Interscience, New York (1975).

[Klei75b] L. Kleinrock and F. A. Tobagi, "Packet Switching in Radio Channels: Part I — Carrier Sense Multiple-Access Modes and Their Throughput-Delay Characteristics," *IEEE Transactions on Communications* **COM-23**(12), pp.1400-1416 (December 1975).

[Klei76] L. Kleinrock, *Queueing Systems, Vol II., Computer Applications*, Wiley-Interscience, New York (1976).

[Konh80] A.G. Konheim, "Mathematical Models for Computer Data Communication," pp. 256-334 in *Case Studies in Mathematical Modeling*, ed. W.E. Boyce, Pitman Advanced Publishing Program, Boston, MA (1980).

[Lam74] S. S. Lam, "Packet Switching in a Multi-Access Broadcast Channel with Application to Satellite Communication in a Computer Network," UCLA-ENG-7429, Computer Science Department, University of California, Los Angeles (April

1974). Ph.D. dissertation.

[Lam80] S. S. Lam, "Packet Broadcast Networks — A Performance Analysis of the R-ALOHA Protocol," *IEEE Transactions on Communications* **COM-29**(7) (July 1980).

[Levy83] H. Levy and L. Kleinrock, "A Queue with Starter: ATS-83010, University of California at Los Angeles (April, 1983). Submitted for publication in Operations Research.

[Levy75] Y. Levy and U. Yechiali, "Utilization of Idle Time in an M/G/1 Queueing System," *Management Science* **22**, pp.202-211 (1975).

[Limb82] J.O. Limb and C. Flores, "Description of Fasnet - A Unidirectional Local-Area Communications Network," *The Bell System Technical Journal* **61**(7), pp.1413-1440 (September 1982).

[Litt61] J. Little, "A Proof of the Queueing Formula $L = \lambda W$," *Operations Research* **9**(2), pp.383-387 (March 1961).

[Mill64] L. Miller, *Alternating Priorities in Multi-class Queues,* Cornell University, Ithaca, New York (1964). Ph.D. Dissertation.

[Mora81] L.F.M. De Moraes Applications to Data Communications Networks, "Message Queueing Delays in Polling Schemes with," UCLA-ENG-8106, Department of System Science, School of Engineering and Applied Science, University of California, Los Angeles, Los Angeles, California (May 1981).

[Nomu78] M. Nomura and K. Tsukamoto, "Traffic Analysis on Polling Systems," *The Transactions of the Institute of Electronics ans Communication Engineering of Japan* **J61-B**(7), pp.600-607 (July 1978). (in Japaneese).

[Robe72] L. G. Roberts, "Extensions of Packet Communication Technology to a Hand Held Personal Terminal," *AFIPS Conference Proceedings, SJCC* **40**, pp.295-298 (1972).

[Rubi81] I. Rubin and L.F. Demoraes, "Polling Schemes for Local Communication Networks," pp. 33.5.1-33.5.7 in *Proceedings International Conference on Communication 1981*, Denver, Colorado (14-18 June 1981).

[Sahi82] V. Sahin, *Analysis of Multihop Packet Radio Networks,* Politechnic Institute of New York (June 1982). Ph.D. Thesis.

[Scho76] M. Scholl, "Multiplexing Techniques for Data Transmission of Packet-Switched Radio Systems," UCLA-ENG-76123, UCLA Computer Science Department, University of California, Los Angeles (December 1976). Ph.D. dissertation.

[Scho83] M. Scholl and L. Kleinrock, "ON the M/G/1 Queue with Rest Periods and Certain Service-Independent Queueing Disciplines," *Operations Research* **31**(4), pp.705-719 (July/August 1983).

[Scho78] F. A. van der Duyn Schouten, "An M/G/1 Queueing Model with Vacation Times," *Zeitschrift Operations Reserach*, pp.95-105 (1978).

[Shan80] J. G. Shanthikumar, "Some Analysis on the Control of Queues Using Level Crossing of Regenerative Processes," *Journal of Applied Probability* **17**, pp.814-821 (1980).

[Sidi83] M. Sidi and A. Segall, "Two interfering Queues in Packet-Radio Networks," *IEEE Transactions on Communications* **COM-31** (Jan 1983).

[Swar80] G. B. Swartz, "Polling in a Loop System," *JACM* **27**(1), pp.42-59 (January 1980).

[Taka83] H. Takagi, L. Kleinrock, and, *A Tutorial on the Analysis of Polling Systems,* University of California at Los Angeles (December 1983).

[Toba74] F. A. Tobagi, "Random Access Techniques for Data Transmission over Packet Switched Radio Networks," UCLA-ENG-7499, Computer Science Department, University of California, Los Angeles (December 1974). Ph.D. Dissertation.

[Toba79] F. A. Tobagi and V. B. Hunt, "Performance Analysis of Carrier Sense Multiple Access with Collision Detection," *Proceedings of the Local Area Communication Network Symposium* (May 1979). also in Computer Networks, November 1980.

[Toba83] F. A. Tobagi and J. M. Brasio, "Throughput Analysis of multihop Packet radio Networks Under Various Channel Access Schemes," *Proceeding of INFCOM'83* (April 1983).

[Welc64] P. D. Welch, "On a Generalized M/G/1 queueing Process in which the First Customer of Each Busy Period Recieves Exceptional Service," *Operations Research* **12**, pp.736-752 (1964).

[Yemi80] Y. Yemini, "On Channel Sharing in Discrete-Time, Multi-Access Broadcast Communications," UCLA-ENG-8227, Computer Science Department, University of California, Los Angeles (September 1980). Ph.D. Dissertation.

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