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GRAPHOIDS: RELATIONAL STRUCTURES WITH GRAPH PROPERTIES

by
Judea Pearl
(Work done in collaboration with Azaria Paz)

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Room 5264 Boelter Hall
3-5 p.m.

SHORT ABSTRACT:

We consider 3-place relations $I(x, S, y)$ where, x and y are two distinct elements, S is any subset of elements not containing x or y, and $I(x, S, y)$ stands for the statement: "Knowing S renders x independent of y". We give sufficient conditions on I for the existence of a (minimal) graph G such that $I(x, S, y)$ can be validated by testing whether S separates x from y in G. These conditions define a GRAPHOID.

EXTENDED ABSTRACT:

Graphs offer a very useful representation for a variety of phenomena. They give vivid visual display for the essential relations in the phenomenon, and provide convenient medium for people to communicate and reason about it. In fact, graph-related concepts are so entrenched in our language that one may wonder whether people can ever reason in any other way, except by tracing links and arrows and paths in some mental representation of concepts and relations. Conversely, if one aspires to use non-numeric logic to mimic human reasoning about knowledge and about beliefs, one should make sure that most derivational steps in that logic would correspond to simple operations on some graphs.

When we deal with a phenomenon where the notion of neighborhood or connectedness is explicit (e.g., family relations, electronic circuits, communication networks etc.) there is no problem in defining the graph which represents the desired characteristics of the phenomenon. However, in modelling conceptual relations such as causation, association, and independence, it is often hard to distinguish direct neighbors from indirect neighbors. The task of constructing a graph representation then becomes more delicate and, once we construct such a graph, it is not always clear which of its properties carries meaningful information about the phenomena under study.

GRAPHOIDS are relational structures which have graphical representations but where the notion of neighborhood is not specified in advance. Rather, what is given explicitly is the relation of "in betweenness". In other words, we can easily test whether any given subset S of elements interferes between elements x and y, but it remains up to us to decide how to connect the elements together in a graph that accounts for these interferences.

A classical example of a Graphoid is the relation of probabilistic independence. For a given probability distribution P, it is fairly easy to verify whether knowing z renders x independent of y. However, P does not dictate which variables should be regarded as direct neighbors. That decision is left to the conceptualizer who must decide which of these dependencies to encode in the graph, and what decoding techniques to use to recover them.

In the case of probabilistic dependencies, we are fortunate to have the theory of Markov-Fields. It tells us how to construct an edge-minimum graph G such that each time we observe a vertex x separated from y by a subset S of vertices, we can be guaranteed that variables x and y are independent given the values of the variables in S. The theory of Graphoids extends this construction to cases where the notion of independence is not given probabilistically or numerically. We now ask what LOGICAL conditions should constrain the qualitative relationship:

$$I(x, S, y) = \text{"knowing } S \text{ renders } x \text{ independent of } y\text{"}$$

so that we can validate it by testing whether S separates x from y in some graph G.

We show that two conditions: (1) weak closure under intersection and (2) weak closure under union, are sufficient to guarantee a unique construction of an edge-minimum graph G that validates $I(x, S, y)$ by vertex separation. These two conditions are chosen as the formal definition of Graphoids, and are shown to yield many of the properties of Markov-Fields.