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8 Appendix

8.1 Proof of Theorem 1

We first prove three lemmas.

Lemma 5. *The z-specific PNS $P(y_x, y'_{x'}|z)$ are bounded as follows:*

$$max\left\{\begin{array}{c} 0 \\ P(y_x|z) - P(y_{x'}|z) \\ P(y|z) - P(y_{x'}|z) \\ P(y_x|z) - P(y|z) \end{array}\right\} \leq \text{z-PNS} \tag{5}$$

$$min\left\{\begin{array}{c} P(y_x|z) \\ P(y'_{x'}|z) \\ P(y,x|z) + P(y',x'|z) \\ P(y_x|z) - P(y_{x'}|z) + \\ +P(y,x'|z) + P(y',x|z) \end{array}\right\} \geq \text{z-PNS} \tag{6}$$

Proof. Since for any three events A, B and C, we have

$$P(A, B|C) \geq max[0, P(A|C) + P(B|C) - 1] \tag{7}$$

therefore, we have

$$\begin{aligned} \text{z-PNS} &\geq max[0, P(y_x|z) + P(y'_{x'}|z) - 1] \\ &= max[0, P(y_x|z) - P(y_{x'}|z)] \end{aligned}$$

Also,

$$\begin{aligned} \text{z-PNS} &= P(y_x, y'_{x'}, x|z) + P(y_x, y'_{x'}, x'|z) \\ &= P(y, y'_{x'}, x|z) + P(y_x, y', x'|z) \qquad (8) \\ &= P(x,y|z) - P(x, y, y_{x'}|z) + P(y_x, y', x'|z) \\ &= P(x,y|z) - P(y, y_{x'}|z) + \\ &\quad P(x', y, y_{x'}|z) + P(y_x, y', x'|z) \\ &= P(x,y|z) - P(y, y_{x'}|z) + \\ &\quad P(x', y|z) + P(y_x, y', x'|z) \\ &= P(y|z) - P(y, y_{x'}|z) + P(x', y', y_x|z) \qquad (9) \\ &= P(y|z) - P(y, y_{x'}|z) + \\ &\quad P(y', y_x|z) - P(x, y', y_x|z) \\ &= P(y|z) - P(y, y_{x'}|z) + \\ &\quad P(y', y_x|z) - P(x, y', y|z) \\ &= P(y|z) - P(y, y_{x'}|z) + P(y', y_x|z) \qquad (10) \end{aligned}$$

By (10),

$$\begin{aligned} \text{z-PNS} &\geq P(y|z) - P(y, y_{x'}|z) \\ &\geq P(y|z) - P(y_{x'}|z) \end{aligned}$$

Also by (10) and (7),

$$\begin{aligned} \text{z-PNS} &\geq P(y|z) - P(y|z) + P(y', y_x|z) \\ &\geq P(y'|z) - P(y'_x|z) \\ &= P(y_x|z) - P(y|z) \end{aligned}$$

Thus, the lower bounds are proved.
And since for any three events A, B and C, we have

$$P(A, B|C) \leq min[P(A|C), P(B|C)] \tag{11}$$

therefore, we have

$$\text{z-PNS} \leq min[P(y_x|z), P(y'_{x'}|z)]$$

Also, by (8),

$$\text{z-PNS} \leq P(x,y|z) + P(x', y'|z)$$

Similarly to (9), we have

$$\begin{aligned} \text{z-PNS} &= P(y'|z) - P(y', y'_x|z) + P(x, y, y'_{x'}|z) \\ &= P(y', y_x|z) + P(x, y, y'_{x'}|z) \\ &= P(y_x|z) - P(y, y_x|z) + P(x, y, y'_{x'}|z) \\ &= P(y_x|z) - P(y, y_x|z) + \\ &\quad P(x,y|z) - P(x, y, y_{x'}|z) \\ &= P(y_x|z) - P(y, y_x|z) + P(x,y|z) - \\ &\quad P(y_{x'}|z) + P(x', y, y_{x'}|z) + \\ &\quad P(x, y', y_{x'}|z) + P(x', y', y_{x'}|z) \\ &= P(y_x|z) - P(y, y_x|z) + \\ &\quad P(x,y|z) - P(y_{x'}|z) + \\ &\quad P(x', y|z) + P(x, y', y_{x'}|z) \\ &= P(y_x|z) - P(y_{x'}|z) + P(x', y|z) + \\ &\quad P(x,y|z) - P(y, y_x|z) + P(x, y', y_{x'}|z) \\ &= P(y_x|z) - P(y_{x'}|z) + P(x', y|z) + \\ &\quad P(x,y|z) - P(x, y, y_x|z) - P(x', y, y_x|z) + \\ &\quad P(x, y'|z) - P(x, y', y'_{x'}|z) \\ &= P(y_x|z) - P(y_{x'}|z) + P(x', y|z) + \\ &\quad P(x, y'|z) - P(x, y', y'_{x'}|z) - P(x', y, y_x|z) \\ &\leq P(y_x|z) - P(y_{x'}|z) + P(x', y|z) + P(x, y'|z) \end{aligned}$$

Thus, the upper bounds are proved. □

Lemma 6.

$$\begin{aligned} &P(y_x, y'_{x'}|z) - P(y'_x, y_{x'}|z) \\ =\; &P(y_x|z) - P(y_{x'}|z) \end{aligned} \tag{12}$$

Proof.

$$\begin{aligned} &P(y_x, y'_{x'}|z) - P(y_{x'}, y'_x|z) \\ =\; &P(y_x, y'_{x'}, x|z) + P(y_x, y'_{x'}, x'|z) - \\ &P(y_{x'}, y'_x, x|z) - P(y_{x'}, y'_x, x'|z) \\ =\; &P(y, y'_{x'}, x|z) + P(y_x, y', x'|z) - \\ &P(y_{x'}, y', x|z) - P(y, y'_x, x'|z) \\ =\; &P(y, y'_{x'}, x|z) - P(y_{x'}, y', x|z) + \\ &P(y_x, y', x'|z) - P(y, y'_x, x'|z) \\ =\; &P(x,y|z) - P(y, y_{x'}, x|z) - P(y_{x'}, y', x|z) + \\ &P(y_x, y', x'|z) + P(y, y_x, x'|z) - P(x', y|z) \\ =\; &P(x,y|z) - P(y_{x'}, x|z) + P(y_x, x'|z) - P(x', y|z) \\ =\; &P(x,y|z) - P(y_{x'}|z) + P(y_{x'}, x'|z) + \\ &P(y_x|z) - P(y_x, x|z) - P(x', y|z) \\ =\; &P(x,y|z) - P(y_{x'}|z) + P(y, x'|z) + \\ &P(y_x|z) - P(y, x|z) - P(x', y|z) \\ =\; &P(y_x|z) - P(y_{x'}|z) \end{aligned}$$

□

Lemma 7. *The counterfactual expression $f(\alpha) = \alpha P(y_x, y'_{x'}|z) - (1-\alpha)P(y_{x'}, y'_x|z)$ for any real number α are bounded as follows.*
Case 1: $\alpha \in (-\infty, 0.5)$

$$max\left\{\begin{array}{c} \alpha P(y_x|z) - (1-\alpha)P(y_{x'}|z) \\ \\ (1-\alpha)P(y_x|z) + \alpha P(y'_{x'}|z) + \alpha - 1 \\ \\ (2\alpha-1)P(y,x|z) + \\ +(2\alpha-1)P(y',x'|z)] + \\ +(1-\alpha)[P(y_x|z) - P(y_{x'}|z)] \\ \\ \alpha[P(y_x|z) - P(y_{x'}|z)] + \\ +(2\alpha-1)P(y,x'|z) + \\ +(2\alpha-1)P(y',x|z) \end{array}\right\}$$

$$\leq f(\alpha) \tag{13}$$

$$min\left\{\begin{array}{c} (1-\alpha)[P(y_x|z) - P(y_{x'}|z)] \\ \\ \alpha[P(y_x|z) - P(y_{x'}|z)] \\ \\ (2\alpha-1)P(y|z) + \\ +(1-\alpha)P(y_x|z) - \alpha P(y_{x'}|z) \\ \\ \alpha P(y_x|z) - \\ -(1-\alpha)P(y_{x'}|z) - (2\alpha-1)P(y|z) \end{array}\right\}$$

$$\geq f(\alpha) \tag{14}$$

Case 2: $\alpha \in [0.5, \infty)$

$$max\left\{\begin{array}{c} (1-\alpha)[P(y_x|z) - P(y_{x'}|z)] \\ \\ \alpha[P(y_x|z) - P(y_{x'}|z)] \\ \\ (2\alpha-1)P(y|z) + \\ +(1-\alpha)P(y_x|z) - \alpha P(y_{x'}|z) \\ \\ \alpha P(y_x|z) - \\ -(1-\alpha)P(y_{x'}|z) - (2\alpha-1)P(y|z) \end{array}\right\}$$

$$\leq f(\alpha) \tag{15}$$

$$min\left\{\begin{array}{c} \alpha P(y_x|z) - (1-\alpha)P(y_{x'}|z) \\ \\ (1-\alpha)P(y_x|z) + \alpha P(y'_{x'}|z) + \alpha - 1 \\ \\ (2\alpha-1)P(y,x|z) + \\ +(2\alpha-1)P(y',x'|z)] + \\ +(1-\alpha)[P(y_x|z) - P(y_{x'}|z)] \\ \\ \alpha[P(y_x|z) - P(y_{x'}|z)] + \\ +(2\alpha-1)P(y,x'|z) + \\ +(2\alpha-1)P(y',x|z) \end{array}\right\}$$

$$\geq f(\alpha) \tag{16}$$

Proof. By lemma 6,

$$\begin{aligned} & f(\alpha) \\ = \ & \alpha P(y_x, y'_{x'}|z) - (1-\alpha)P(y_{x'}, y'_x|z) \\ = \ & \alpha P(y_x, y'_{x'}|z) - \\ & (1-\alpha)(P(y_x, y'_{x'}|z) - P(y_x|z) + P(y_{x'}|z)) \\ = \ & (2\alpha-1)P(y_x, y'_{x'}|z) + \\ & (1-\alpha)(P(y_x|z) - P(y_{x'}|z)) \end{aligned} \tag{17}$$

By lemma 5, substituting (5) and (6) into (17), case 1 and 2 in lemma 7 hold. □

Now, let's prove theorem 1.

Proof.

$$\begin{aligned} & f(\beta, \gamma, \theta, \delta) \\ = \ & \beta P(y_x, y'_{x'}|z) + \gamma P(y_x, y_{x'}|z) + \\ & \theta P(y'_x, y'_{x'}|z) + \delta P(y'_x, y_{x'}|z) \\ = \ & \beta P(y_x, y'_{x'}|z) + \gamma[P(y_x|z) - P(y_x, y'_{x'}|z)] + \\ & \theta[P(y'_{x'}) - P(y_x, y'_{x'}|z)] + \delta P(y'_x, y_{x'}|z) \\ = \ & \gamma P(y_x|z) + \theta P(y'_{x'}|z) + \\ & (\beta - \gamma - \theta)P(y_x, y'_{x'}|z) - (-\delta)P(y'_x, y_{x'}|z) \end{aligned} \tag{18}$$

By lemma 7, let $\alpha = \frac{\beta-\gamma-\theta}{\beta-\gamma-\theta-\delta}$, substituting (13) to (16) into (18), theorem 1 hold. □

8.2 Proof of Theorem 4

Lemma 8. *If Y is monotonic relative to X, z-specific $PNS = P(y_x, y'_{x'}|z)$ is identifiable whenever the causal effects $P(y_x|z)$ and $P(y_{x'}|z)$ are identifiable:*

$$\begin{aligned} PNS &= P(y_x, y'_{x'}|z) \\ &= P(y_x|z) - P(y_{x'}|z). \end{aligned}$$

Proof. Since $y_{x'}$ and $y'_{x'}$ are complementary, so $y_{x'} \vee y'_{x'}$ = true, therefore, we have

$$y_x = y_x \wedge (y_{x'} \vee y'_{x'}) = (y_x \wedge y_{x'}) \vee (y_x \wedge y'_{x'}) \tag{19}$$

Similarly,

$$\begin{aligned} y_{x'} &= y_{x'} \wedge (y_x \vee y'_x) \\ &= (y_{x'} \wedge y_x) \vee (y_{x'} \wedge y'_x) \\ &= y_{x'} \wedge y_x \end{aligned} \tag{20}$$

Since monotonicity entails that $y_{x'} \wedge y'_x$ = false.
Substituting (20) into (19) yields

$$y_x = y_{x'} \vee (y_x \wedge y'_{x'})$$

Thus, for any z, we have,

$$y_x \wedge z = (y_{x'} \wedge z) \vee (y_x \wedge y'_{x'} \wedge z) \tag{21}$$

Taking the probability of (21) and using the disjointness of $y_{x'}$ and $y'_{x'}$, we obtain

$$P(y_x, z) = P(y_{x'}, z) + P(y_x, y'_{x'}, z)$$

Therefore,

$$P(y_x|z) = P(y_{x'}|z) + P(y_x, y'_{x'}|z)$$

or

$$P(y_x, y'_{x'}|z) = P(y_x|z) - P(y_{x'}|z) \tag{22}$$

□

Now, let's prove Theorem 4.

Proof.

$$\begin{aligned}
& f(\beta, \gamma, \theta, \delta) \\
= \ & \beta P(y_x, y'_{x'}|z) + \gamma P(y_x, y_{x'}|z) + \\
& \theta P(y'_x, y'_{x'}|z) + \delta P(y'_x, y_{x'}|z) \\
= \ & \beta[P(y_x|z) - P(y_x, y_{x'}|z)] + \\
& \gamma[P(y_{x'}|z) - P(y'_x, y_{x'}|z)] + \\
& \theta[P(y'_x|z) - P(y'_x, y_{x'}|z)] + \delta P(y'_x, y_{x'}|z) \\
= \ & \beta[P(y_x|z) - P(y_{x'}|z) + P(y'_x, y_{x'}|z)] + \\
& \gamma[P(y_{x'}|z) - P(y'_x, y_{x'}|z)] + \\
& \theta[P(y'_x|z) - P(y'_x, y_{x'}|z)] + \delta P(y'_x, y_{x'}|z) \\
= \ & \beta P(y_x|z) + (\gamma - \beta) P(y_{x'}|z) + \theta P(y'_x|z) + \\
& (\beta + \delta - \gamma - \theta) P(y'_x, y_{x'}|z)
\end{aligned}$$

Thus, with $\beta + \delta = \gamma + \theta$, theorem 4 hold.
Also if monotonicity, we have,

$$P(y_{x'}, y'_x|z) = 0 \tag{23}$$

By lemma 8, substituting (23) and (22) into (18), theorem 4 holds. □