

Generalized Instrumental Variables

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Abstract

This paper concerns the assessment of direct causal effects from a combination of: (i) non-experimental data, and (ii) qualitative domain knowledge. Domain knowledge is encoded in the form of a directed acyclic graph (DAG), in which all interactions are assumed linear, and some variables are presumed to be unobserved. We provide a generalization of the well-known method of Instrumental Variables, which allows its application to models with few conditional independences.

1 Introduction

This paper explores the feasibility of inferring linear cause-effect relationships from various combinations of data and theoretical assumptions. The assumptions are represented in the form of an acyclic causal diagram which contains both arrows and bi-directed arcs [9, 10]. The arrows represent the potential existence of direct causal relationships between the corresponding variables, and the bi-directed arcs represent spurious correlations due to unmeasured common causes. All interactions among variables are assumed to be linear. Our task is to decide whether the assumptions represented in the diagram are sufficient for assessing the strength of causal effects from non-experimental data, and, if sufficiency is proven, to express the target causal effect in terms of estimable quantities.

This decision problem has been tackled in the past half century, primarily by econometricians and social scientists, under the rubric “The Identification Problem” [6] – it is still unsolved. Certain restricted classes of models are nevertheless known to be identifiable, and these are often assumed by social scientists as a matter of convenience or convention [5]. A hierarchy of three such classes is given in [7]: (1) no bidirected arcs, (2) bidirected arcs restricted to root variables,

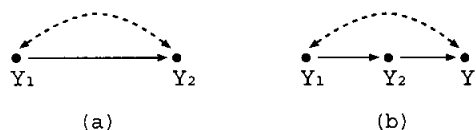


Figure 1: (a) a “bow-pattern”, and (b) a bow-free model

and (3) bidirected arcs restricted to variables that are not connected through directed paths.

Recently [4], we have shown that the identification of the entire model is ensured if variables standing in direct causal relationship (i.e., variables connected by arrows in the diagram) do not have correlated errors; no restrictions need to be imposed on errors associated with indirect causes. This class of models was called “bow-free”, since their associated causal diagrams are free of any “bow pattern” [10] (see Figure 1).

Most existing conditions for Identification in general models are based on the concept of Instrumental Variables (IV) [11], [2]. IV methods take advantage of conditional independence relations implied by the model to prove the Identification of specific causal-effects. When the model is not rich in conditional independences, these methods are not much informative. In [3], we proposed a new graphical criterion for Identification which does not make direct use of conditional independence, and thus can be successfully applied to models in which IV methods would fail.

In this paper, we provide an important generalization of the method of Instrumental Variables that makes it less sensitive to the independence relations implied by the model.

2 Linear Models and Identification

An equation $Y = \beta X + e$ encodes two distinct assumptions: (1) the possible existence of (direct) causal influence of X on Y ; and, (2) the absence of causal in-

$$\begin{aligned}
Z &= e_1 \\
W &= e_2 \\
X &= aZ + e_3 \\
Y &= bW + cX + e_4 \\
Cov(e_1, e_2) &= \alpha \neq 0 \\
Cov(e_2, e_3) &= \beta \neq 0 \\
Cov(e_3, e_4) &= \gamma \neq 0
\end{aligned}$$

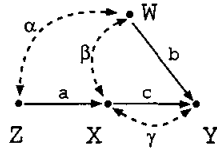


Figure 2: A simple linear model and its causal diagram

fluence on Y of any variable that does not appear on the right-hand side of the equation. The parameter β quantifies the (direct) causal effect of X on Y . That is, the equation claims that a unit increase in X would result in β units increase of Y , assuming that everything else remains the same. The variable e is called an "error" or "disturbance"; it represents unobserved background factors that the modeler decides to keep unexplained.

A linear model for a set of random variables $\mathbf{Y} = \{Y_1, \dots, Y_n\}$ is defined formally by a set of equations of the form

$$Y_j = \sum_i c_{ji} Y_i + e_j \quad , j = 1, \dots, n$$

and an error variance/covariance matrix Ψ , i.e., $[\Psi_{ij}] = Cov(e_i, e_j)$. The error terms e_j are assumed to have normal distribution with zero mean.

The equations and the pairs of error-terms (e_i, e_j) with non-zero correlation define the structure of the model. The model structure can be represented by a directed graph, called causal diagram, in which the set of nodes is defined by the variables Y_1, \dots, Y_n , and there is a directed edge from Y_i to Y_j if the coefficient of Y_i in the equation for Y_j is different from zero. Additionally, if error-terms e_i and e_j have non-zero correlation, we add a (dashed) bidirected edge between Y_i and Y_j . Figure 2 shows a model with the respective causal diagram.

The structural parameters of the model, denoted by θ , are the coefficients c_{ij} , and the non-zero entries of the error covariance matrix Ψ . In this work, we consider only recursive models, that is, $c_{ji} = 0$ for $i \geq j$.

Fixing the model structure and assigning values to the parameters θ , the model determines a unique covariance matrix Σ over the observed variables $\{Y_1, \dots, Y_n\}$, given by (see [1], page 85)

$$\Sigma(\theta) = (I - C)^{-1} \Psi [(I - C)^{-1}]^T \quad (1)$$

where C is the matrix of coefficients c_{ji} .

Conversely, in the Identification problem, after fixing the structure of the model, one attempts to solve for

θ in terms of the observed covariance Σ . This is not always possible. In some cases, no parametrization of the model could be compatible with a given Σ . In other cases, the structure of the model may permit several distinct solutions for the parameters. In these cases, the model is called *nonidentified*.

Sometimes, although the model is nonidentifiable, some parameters may be uniquely determined by the given assumptions and data. Whenever this is the case, the specific parameters are *identified*.

Finally, since the conditions we seek involve the structure of the model alone, and do not depend on the numerical values of parameters θ , we insist only on having identification almost everywhere, allowing few pathological exceptions. The concept of identification almost everywhere is formalized in section 6.

3 Graph Background

Definition 1 A path in a graph is a sequence of edges (directed or bidirected) such that each edge starts in the node ending the preceding edge. A directed path is a path composed only by directed edges, all oriented in the same direction. Node X is a descendent of node Y if there is a directed path from Y to X . Node Z is a collider in a path p if there is a pair of consecutive edges in p such that both edges are oriented toward Z (e.g., $\dots \rightarrow Z \leftarrow \dots$).

Let p be a path between X and Y , and let Z be an intermediate variable in p . We denote by $p[X \sim Z]$ the subpath of p consisting of the edges between X and Z .

Definition 2 (*d-separation*)

A set of nodes \mathbf{Z} *d-separates* X from Y in a graph, if Z blocks every path between X and Y . A path p is blocked by a set \mathbf{Z} (possibly empty) if one of the following holds:

- (i) p contains at least one non-collider that is in \mathbf{Z} ;
- (ii) p contains at least one collider that is outside \mathbf{Z} and has no descendant in \mathbf{Z} .

4 Instrumental Variable Methods

The traditional definition qualifies a variable Z as instrumental, relative to a cause X and effect Y if [10]:

1. Z is independent of all error terms that have an influence on Y which is not mediated by X ;
2. Z is not independent of X .

The intuition behind this definition is that all correlation between Z and Y must be intermediated by X .

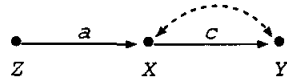


Figure 3: Typical Instrumental Variable

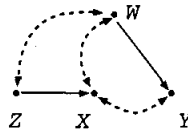


Figure 4: Conditional IV Examples

If we can find Z with these properties, then the causal effect of X on Y , denoted by c , is identified and given by $c = \sigma_{ZY} / \sigma_{ZX}$.

Figure 3 shows a typical example of an instrumental variable. It is easy to verify that variable Z satisfy properties (1) and (2) in this model.

A generalization of the IV method is offered through the use of conditional IV's. A conditional IV is a variable Z that may not have properties (1) and (2), but there is a conditioning set W which makes it happen. When such pair (Z, W) is found, the causal effect of X on Y is identified and given by $c = \sigma_{ZY.W} / \sigma_{ZX.W}$.

[11] provides the following equivalent graphical criterion for conditional IV's, based on the concept of d-separation:

1. W contains only non-descendants of Y ;
2. W d-separates Z from Y in the subgraph G_c obtained by removing edge $X \rightarrow Y$ from G ;
3. W does not d-separate Z from X in G_c .

As an example of the application of this criterion, Figure 4 show the graph obtained by removing edge $X \rightarrow Y$ from the model of Figure 2. After conditioning on variable W , Z becomes d-separated from Y but not from X . Thus, parameter c is identified.

5 Instrumental Sets

Although very useful, the method of conditional IV's has some limitations. As an example, Figure (5a) shows a simple model in which the method cannot be applied. In this model, variables Z_1 and Z_2 do not qualify as IV's with respect to either c_1 or c_2 . Also, there is no conditioning set which makes it happen. Therefore, the conditional IV method fails, despite the fact that the model is completely identified.

Following the ideas stated in the graphical criterion for conditional IV's, we show in Figure (5b) the graph

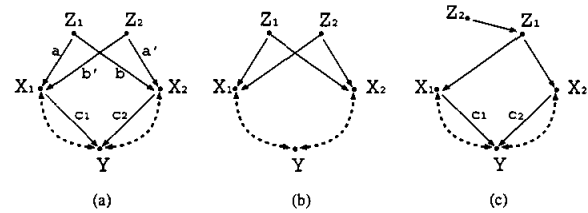


Figure 5: Simultaneous use of two IVs

obtained by removing edges $X_1 \rightarrow Y$ and $X_2 \rightarrow Y$ from the model. Note that in this graph, Z_1 and Z_2 satisfy the graphical conditions for a conditional IV. Intuitively, if we could use both Z_1 and Z_2 together as instrumental variables, we would be able to identify parameters c_1 and c_2 . This motivates the following informal definition:

A set of variables $Z = \{Z_1, \dots, Z_k\}$ is called an *instrumental set* relative to a set of causes $X = \{X_1, \dots, X_n\}$ and an effect Y if:

1. Each $Z_i \in Z$ is independent of all error terms that have an influence on Y which is not mediated by some $X_j \in X$;
2. Each $Z_i \in Z$ is not independent of the respective $X_i \in X$, for appropriate enumerations of Z and X ;
3. The set Z is not redundant with respect to Y . That is, for any $Z_i \in Z$ we cannot explain the correlation between Z_i and Y by correlations between Z_i and $Z - \{Z_i\}$, and correlations between $Z - \{Z_i\}$ and Y .

Properties 1 and 2 above are similar to the ones in the definition of Instrumental Variables, and property 3 is required when using more than one instrument. To see why we need the extra condition, let us consider the model in Figure (5c). In this example, the correlation between Z_2 and Y is given by the product of the correlation between Z_2 and Z_1 and the correlation between Z_1 and Y . That is, Z_2 does not give additional information once we already have Z_1 . In fact, using Z_1 and Z_2 as instruments we cannot obtain the identification of the causal effects of X_1 and X_2 on Y .

Now, we give a precise definition of instrumental sets using graphical conditions. Fix a variable Y and let $X = \{X_1, \dots, X_k\}$ be a set of direct causes of Y .

Definition 3 The set $Z = \{Z_1, \dots, Z_n\}$ is said to be an *Instrumental Set* relative to X and Y if we can find triples $(Z_1, W_1, p_1), \dots, (Z_n, W_n, p_n)$, such that:

- (i) For $i = 1, \dots, n$, Z_i and the elements of W_i

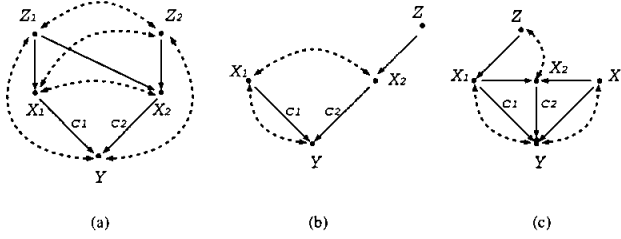


Figure 6: More examples of Instrumental Sets

are non-descendants of Y ; and p_i is an unblocked path between Z_i and Y including edge $X_i \rightarrow Y$.

- (ii) Let \bar{G} be the causal graph obtained from G by deleting edges $X_1 \rightarrow Y, \dots, X_n \rightarrow Y$. Then, \mathbf{W}_i d -separates Z_i from Y in \bar{G} ; but \mathbf{W}_i does not block path p_i ;
- (iii) For $1 \leq i < j \leq n$, variable Z_j does not appear in path p_i ; and, if paths p_i and p_j have a common variable V , then both $p_i[V \sim Y]$ and $p_j[Z_j \sim V]$ point to V .

Next, we state the main result of this paper.

Theorem 1 If $\mathbf{Z} = \{Z_1, \dots, Z_n\}$ is an instrumental set relative to causes $\mathbf{X} = \{X_1, \dots, X_n\}$ and effect Y , then the parameters of edges $X_1 \rightarrow Y, \dots, X_n \rightarrow Y$ are identified almost everywhere, and can be computed by solving a system of linear equations.

Figure 6 shows more examples in which the method of conditional IV's fails and our new criterion is able to prove the identification of parameters c_i 's. In particular, model (a) is a bow-free model, and thus is completely identifiable. Model (b) illustrates an interesting case in which variable X_2 is used as the instrument for $X_1 \rightarrow Y$, while Z is the instrument for $X_2 \rightarrow Y$. Finally, in model (c) we have an example in which the parameter of edge $X_3 \rightarrow Y$ is nonidentifiable, and still the method can prove the identification of c_1 and c_2 .

The remaining of the paper is dedicated to the proof of Theorem 1.

6 Preliminary Results

6.1 Identification Almost Everywhere

Let h denote the total number of parameters in model G . Then, each vector $\theta \in \mathbb{R}^h$ defines a parametrization of the model. For each parametrization θ , model G generates a unique covariance matrix $\Sigma(\theta)$. Let $\theta(\lambda_1, \dots, \lambda_n)$ denote the vector of values assigned by θ to parameters $\lambda_1, \dots, \lambda_n$.

Parameters $\lambda_1, \dots, \lambda_n$, are identified almost everywhere if $\Sigma(\theta) = \Sigma(\theta')$ implies $\theta(\lambda_1, \dots, \lambda_n) = \theta'(\lambda_1, \dots, \lambda_n)$, except when θ resides on a set of Lebesgue measure zero.

6.2 Wright's Method of Path Coefficients

Here, we describe an important result introduced by Sewall Wright [12], which is extensively explored in the proof.

Given variables X and Y in a recursive linear model, the correlation coefficient of X and Y , denoted ρ_{XY} , can be expressed as a polynomial on the parameters of the model. More precisely,

$$\rho_{Z,Y} = \sum_{\text{paths } p_i} T(p_i) \quad (2)$$

where term $T(p_i)$ represents the multiplication of the parameters of edges along path p_i , and the summation ranges over all unblocked paths between X and Y . For this equality to hold, the variables in the model must be standardized (variance equal to 1) and have zero mean. However, if this is not the case, a simple transformation can put the model in this form [13]. We refer to Eq.(2) as Wright's Equation for X and Y .

Wright's method of path coefficients [12] consists in forming Eq.(2) for each pair of variables in the model, and solving for the parameters in terms of the correlations among the variables. Whenever there is a unique solution for a parameter λ , this parameter is identified.

We can use this method to study the identification of the parameters in the model of Figure 5. From the equations for ρ_{Y_1, Y_5} and ρ_{Y_2, Y_5} we can see that parameters c_1 and c_2 are identified if and only if

$$\text{Det} \begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \neq 0$$

6.3 Partial Correlation Lemma

Next lemma provides a convenient expression for the partial correlation coefficient of Y_1 and Y_2 , given Y_3, \dots, Y_n , denoted $\rho_{12.3\dots n}$. The proof of the lemma is given in the appendix.

Lemma 1 The partial correlation $\rho_{12.3\dots n}$ can be expressed as the ratio:

$$\rho_{12.3\dots n} = \frac{\phi(1, 2, \dots, n)}{\psi(1, 3, \dots, n) \cdot \psi(2, 3, \dots, n)} \quad (3)$$

where ϕ and ψ are functions of the correlations among Y_1, Y_2, \dots, Y_n , satisfying the following conditions:

- (i) $\phi(1, 2, \dots, n) = \phi(2, 1, \dots, n)$.

- (ii) $\phi(1, 2, \dots, n)$ is linear on the correlations $\rho_{12}, \rho_{32}, \dots, \rho_{n2}$, with no constant term.
- (iii) The coefficients of $\rho_{12}, \rho_{32}, \dots, \rho_{n2}$, in $\phi(1, 2, \dots, n)$ are polynomials on the correlations among the variables Y_1, Y_3, \dots, Y_n . Moreover, the coefficient of ρ_{12} has the constant term equal to 1, and the coefficients of $\rho_{32}, \dots, \rho_{n2}$, are linear on the correlations $\rho_{13}, \rho_{14}, \dots, \rho_{1n}$, with no constant term.
- (iv) $(\psi(i_1, \dots, i_{n-1}))^2$, is a polynomial on the correlations among the variables $Y_{i_1}, \dots, Y_{i_{n-1}}$, with constant term equal to 1.

6.4 Path Lemmas

The following lemmas explore some consequences of the conditions in the definition of Instrumental Sets.

Lemma 2 *W.l.o.g., we may assume that, for $1 \leq i < j \leq n$, paths p_i and p_j do not have any common variable other than (possibly) Z_i .*

Proof: Assume that paths p_i and p_j have some variables in common, different from Z_i . Let V be the closest variable to X_i in path p_i which also belongs to path p_j .

We show that after replacing triple (Z_i, \mathbf{W}_i, p_i) by triple $(V, \mathbf{W}_i, p_i[V \sim Y])$, conditions (i) – (iii) still hold.

It follows from condition (iii) that subpath $p_i[V \sim Y]$ must point to V . Since p_i is unblocked, subpath $p_i[Z_i \sim V]$ must be a directed path from V to Z_i .

Now, variable V cannot be a descendent of Y , because $p_i[Z_i \sim V]$ is a directed path from V to Z_i , and Z_i is a non-descendent of Y . Thus, condition (i) still holds.

Consider the causal graph \overline{G} . Assume that there exists a path p between V and Y witnessing that \mathbf{W}_i does not d-separate V from Y in \overline{G} . Since $p_i[Z_i \sim V]$ is a directed path from V to Z_i , we can always find another path witnessing that \mathbf{W}_i does not d-separate Z_i from Y in \overline{G} (for example, if p and $p_i[Z_i \sim V]$ do not have any variable in common other than V , then we can just take their concatenation). But this is a contradiction, and thus it is easy to see that condition (ii) still holds.

Condition (iii) follows from the fact that $p_i[V \sim Y]$ and $p_j[Z_j \sim V]$ point to V . \square

In the following, we assume that the conditions of lemma 2 hold.

Lemma 3 *For all $1 \leq i \leq n$, there exists no unblocked path between Z_i and Y , different from p_i , which includes edge $X_i \rightarrow Y$ and is composed only by edges from p_1, \dots, p_i .*

Proof: Let p be an unblocked path between Z_i and Y , different from p_i , and assume that p is composed only by edges from p_1, \dots, p_i .

According to condition (iii), if Z_i appears in some path p_j , with $j \neq i$, then it must be that $j > i$. Thus, p must start with some edges of p_i .

Since p is different from p_i , it must contain at least one edge from p_1, \dots, p_{i-1} . Let (V_1, V_2) denote the first edge in p which does not belong to p_i .

From lemma 2, it follows that variable V_1 must be a Z_k for some $k < i$, and by condition (iii), both subpath $p[Z_i \sim V_1]$ and edge (V_1, V_2) must point to V_1 . But this implies that p is blocked by V_1 , which contradicts our assumptions. \square

The proofs for the next two lemmas are very similar to the previous one, and so are omitted.

Lemma 4 *For all $1 \leq i \leq n$, there is no unblocked path between Z_i and some W_i , composed only by edges from p_1, \dots, p_i .*

Lemma 5 *For all $1 \leq i \leq n$, there is no unblocked path between Z_i and Y including edge $X_j \rightarrow Y$, with $j < i$, composed only by edges from p_1, \dots, p_i .*

7 Proof of Theorem 1

7.1 Notation

Fix a variable Y in the model. Let $\mathbf{X} = \{X_1, \dots, X_k\}$ be the set of all non-descendents of Y which are connected to Y by an edge (directed, bidirected, or both). Define the following set of edges with an arrowhead at Y :

$$Inc(Y) = \{(X_i, Y) : X_i \in \mathbf{X}\}$$

Note that for some $X_i \in \mathbf{X}$ there may be more than one edge between X_i and Y (one directed and one bidirected). Thus, $|Inc(Y)| \geq |\mathbf{X}|$. Let $\lambda_1, \dots, \lambda_m$, $m \geq k$, denote the parameters of the edges in $Inc(Y)$.

It follows that edges $X_1 \rightarrow Y, \dots, X_n \rightarrow Y$, belong to $Inc(Y)$, because X_1, \dots, X_n , are clearly non-descendents of Y . W.l.o.g., let λ_i be the parameter of edge $X_i \rightarrow Y$, $1 \leq i \leq n$, and let $\lambda_{n+1}, \dots, \lambda_m$ be the parameters of the remaining edges in $Inc(Y)$.

Let Z be any non-descendent of Y . Wright's equation for the pair (Z, Y) , is given by

$$\rho_{Z,Y} = \sum_{\text{paths } p_i} T(p_i) \tag{4}$$

where each term $T(p_i)$ corresponds to an unblocked path between Z and Y . Next lemma proves a property of such paths.

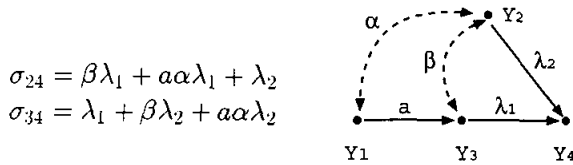


Figure 7: Wright's equations

Lemma 6 Let Y be a variable in a recursive model, and let Z be a non-descendent of Y . Then, any unblocked path between Z and Y must include exactly one edge from $\text{Inc}(Y)$.

Lemma 6 allows us to write Eq. (4) as

$$\rho_{Z,Y} = \sum_{j=1}^m a_j \cdot \lambda_j \quad (5)$$

Thus, the correlation between Z and Y can be expressed as a linear function of the parameters $\lambda_1, \dots, \lambda_m$, with no constant term. Figure 7 shows an example of those equations for a simple model.

7.2 Basic Linear Equations

Consider a triple (Z_i, \mathbf{W}_i, p_i) , and let $\mathbf{W}_i = \{W_{i_1}, \dots, W_{i_k}\}$ ¹. From lemma 1, we can express the partial correlation of Z_i and Y given \mathbf{W}_i as:

$$\rho_{Z_i Y, \mathbf{W}_i} = \frac{\phi_i(Z_i, Y, W_{i_1}, \dots, W_{i_k})}{\psi_i(Z_i, W_{i_1}, \dots, W_{i_k}) \cdot \psi_i(Y, W_{i_1}, \dots, W_{i_k})} \quad (6)$$

where function ϕ_i is linear on the correlations $\rho_{Z_i Y}$, $\rho_{W_{i_1} Y}$, \dots , $\rho_{W_{i_k} Y}$, and ψ_i is a function of the correlations among the variables given as arguments. We abbreviate $\phi_i(Z_i, Y, W_{i_1}, \dots, W_{i_k})$ by $\phi_i(Z_i, Y, \mathbf{W}_i)$, and $\psi_i(W_{i_1}, \dots, W_{i_k})$ by $\psi_i(\mathbf{W}_i)$.

We have seen that the correlations $\rho_{Z_i Y}$, $\rho_{W_{i_1} Y}$, \dots , $\rho_{W_{i_k} Y}$, can be expressed as linear functions of the parameters $\lambda_1, \dots, \lambda_m$. Since ϕ_i is linear on these correlations, it follows that we can express ϕ_i as a linear function of the parameters $\lambda_1, \dots, \lambda_m$.

Formally, by lemma 1, $\phi_i(Z_i, Y, \mathbf{W}_i)$ can be written as:

$$\phi_i(Z_i, Y, \mathbf{W}_i) = b_{i_0} \rho_{Z_i Y} + b_{i_1} \rho_{W_{i_1} Y} + \dots + b_{i_k} \rho_{W_{i_k} Y} \quad (7)$$

Also, for each $V_j \in \{Z_i\} \cup \mathbf{W}_i$, we can write:

$$\rho_{V_j Y} = a_{i,j,1} \lambda_1 + \dots + a_{i,j,m} \lambda_m \quad (8)$$

¹To simplify the notation, we assume that $|\mathbf{W}_i| = k$, for $i = 1, \dots, n$

Replacing each correlation in Eq.(7) by the expression given by Eq. (8), we obtain

$$\phi_i(Z_i, Y, \mathbf{W}_i) = q_{i1} \lambda_1 + \dots + q_{im} \lambda_m \quad (9)$$

where the coefficients q_{il} 's are given by:

$$q_{il} = \sum_{j=0}^k b_{i,j} a_{i,j,l}, \quad l = 1, \dots, m \quad (10)$$

Lemma 7 The coefficients $q_{i,n+1}, \dots, q_{im}$ in Eq. (9) are identically zero.

Proof: The fact that \mathbf{W}_i d-separates Z_i from Y in \overline{G} , implies that $\rho_{Z_i Y, \mathbf{W}_i} = 0$ in any probability distribution compatible with \overline{G} ([10], pg. 142). Thus, $\phi_i(Z_i, Y, \mathbf{W}_i)$ must vanish when evaluated in \overline{G} . But this implies that the coefficient of each of the λ_i 's in Eq. (9) must be identically zero.

Now, we show that the only difference between evaluations of $\phi_i(Z_i, Y, \mathbf{W}_i)$ on the causal graphs \overline{G} and G , consists on the coefficients of parameters $\lambda_1, \dots, \lambda_n$.

First, observe that coefficients b_{i_0}, \dots, b_{i_k} are polynomials on the correlations among the variables $Z_i, W_{i_1}, \dots, W_{i_k}$. Thus, they only depend on the unblocked paths between such variables in the causal graph. However, the insertion of edges $X_1 \rightarrow Y, \dots, X_n \rightarrow Y$, in \overline{G} does not create any new unblocked path between any pair of $Z_i, W_{i_1}, \dots, W_{i_k}$ (and obviously does not eliminate any existing one). Hence, the coefficients b_{i_0}, \dots, b_{i_k} have exactly the same value in the evaluations of $\phi_i(Z_i, Y, \mathbf{W}_i)$ on \overline{G} and G .

Now, let λ_l be such that $l > n$, and let $V_j \in \{Z_i\} \cup \mathbf{W}_i$. Note that the insertion of edges $X_1 \rightarrow Y, \dots, X_n \rightarrow Y$, in \overline{G} does not create any new unblocked path between V_j and Y including the edge whose parameter is λ_l (and does not eliminate any existing one). Hence, coefficients $a_{i,j,l}$, $j = 0, \dots, k$, have exactly the same value on \overline{G} and G .

From the two previous facts, we conclude that, for $l > n$, the coefficient of λ_l in the evaluations of $\phi_i(Z_i, Y, \mathbf{W}_i)$ on \overline{G} and G have exactly the same value, namely zero. Next, we argue that $\phi_i(Z_i, Y, \mathbf{W}_i)$ does not vanish when evaluated on G .

Finally, let λ_l be such that $l \leq n$, and let $V_j \in \{Z_i\} \cup \mathbf{W}_i$. Note that there is no unblocked path between V_j and Y in \overline{G} including edge $X_l \rightarrow Y$, because this edge does not exist in \overline{G} . Hence, the coefficient of λ_l in the expression for the correlation $\rho_{V_j Y}$ on \overline{G} must be zero.

On the other hand, the coefficient of λ_l in the same expression on G is not necessarily zero. In fact, it follows from the conditions in the definition of Instrumental sets that, for $l = i$, the coefficient of λ_i contains the term $T(p_i)$. \square

From lemma 7, we get that $\phi_i(Z_i, Y, \mathbf{W}_i)$ is a linear function only on the parameters $\lambda_1, \dots, \lambda_n$.

7.3 System of Equations Φ

Rewriting Eq.(6) for each triple (Z_i, \mathbf{W}_i, p_i) , we obtain the following system of linear equations on the parameters $\lambda_1, \dots, \lambda_n$:

$$\Phi = \begin{cases} \phi_1(Z_1, Y, \mathbf{W}_1) = \rho_{Z_1 Y, \mathbf{W}_1} \cdot \psi_1(Z_1, \mathbf{W}_1) \cdot \psi_1(Y, \mathbf{W}_1) \\ \dots \\ \phi_n(Z_n, Y, \mathbf{W}_n) = \rho_{Z_n Y, \mathbf{W}_n} \cdot \psi_n(Z_n, \mathbf{W}_n) \cdot \psi_n(Y, \mathbf{W}_n) \end{cases}$$

where the terms on the right-hand side can be computed from the correlations among the variables $Y, Z_i, W_{i_1}, \dots, W_{i_k}$, estimated from data.

Our goal is to show that Φ can be solved uniquely for the λ_i 's, and so prove the identification of $\lambda_1, \dots, \lambda_n$. Next lemma proves an important result in this direction. Let Q denote the matrix of coefficients of Φ .

Lemma 8 *Det(Q) is a non-trivial polynomial on the parameters of the model.*

Proof: From Eq.(10), we get that each entry q_{il} of Q is given by

$$q_{il} = \sum_{j=0}^k b_{ij} \cdot a_{ij,l}$$

where b_{ij} is the coefficient of $\rho_{W_{i_j} Y}$ (or $\rho_{Z_i Y}$, if $j = 0$), in the linear expression for $\phi_i(Z_i, Y, \mathbf{W}_i)$ in terms of correlations (see Eq.(7)); and $a_{ij,l}$ is the coefficient of λ_l in the expression for the correlation $\rho_{W_{i_j} Y}$ in terms of the parameters $\lambda_1, \dots, \lambda_m$ (see Eq.(8)).

From property (iii) of lemma 1, we get that b_{i_0} has constant term equal to 1. Thus, we can write $b_{i_0} = 1 + \hat{b}_{i_0}$, where \hat{b}_{i_0} represent the remaining terms of b_{i_0} .

Also, from condition (i) of Theorem 1, it follows that $a_{i_0 i}$ contains term $T(p_i)$. Thus, we can write $a_{i_0 i} = T(p_i) + \hat{a}_{i_0 i}$, where $\hat{a}_{i_0 i}$ represents all the remaining terms of $a_{i_0 i}$.

Hence, a diagonal entry q_{ii} of Q , can be written as

$$q_{ii} = T(p_i)[1 + \hat{b}_{i_0}] + \hat{a}_{i_0 i} \cdot b_{i_0} + \sum_{j=1}^k b_{ij} \cdot a_{ij,i} \quad (11)$$

Now, the determinant of Q is defined as the weighted sum, for all permutations π of $\langle 1, \dots, n \rangle$, of the product of the entries selected by π (entry q_{il} is selected by

permutation π if the i^{th} element of π is l), where the weights are 1 or (-1) , depending on the parity of the permutation. Then, it is easy to see that the term

$$T^* = \prod_{j=1}^n T(p_j)$$

appears in the product of permutation $\pi = \langle 1, \dots, n \rangle$, which selects all the diagonal entries of Q .

We prove that $\det(Q)$ does not vanish by showing that T^* appears only once in the product of permutation $\langle 1, \dots, n \rangle$, and that T^* does not appear in the product of any other permutation.

Before proving those facts, note that, from the conditions of lemma 2, for $1 \leq i < j \leq n$, paths p_i and p_j have no edge in common. Thus, every factor of T^* is distinct from each other.

Proposition: Term T^* appears only once in the product of permutation $\langle 1, \dots, n \rangle$.

Proof: Let τ be a term in the product of permutation $\langle 1, \dots, n \rangle$. Then, τ has one factor corresponding to each diagonal entry of Q .

A diagonal entry q_{ii} of Q can be expressed as a sum of three terms (see Eq.(11)).

Let i be such that for all $l > i$, the factor of τ corresponding to entry q_{il} comes from the first term of q_{il} (i.e., $T(p_l)[1 + \hat{b}_{l_0}]$).

Assume that the factor of τ corresponding to entry q_{ii} comes from the second term of q_{ii} (i.e., $\hat{a}_{i_0 i} \cdot b_{i_0}$). Recall that each term in $\hat{a}_{i_0 i}$ corresponds to an unblocked path between Z_i and Y , different from p_i , including edge $X_i \rightarrow Y$. However, from lemma 3, any such path must include either an edge which does not belong to any of p_1, \dots, p_n , or an edge which appears in some of p_{i+1}, \dots, p_n . In the first case, it is easy to see that τ must have a factor which does not appear in T^* . In the second, the parameter of an edge of some $p_l, l > i$, must appear twice as a factor of τ , while it appears only once in T^* . Hence, τ and T^* are distinct terms.

Now, assume that the factor of τ corresponding to entry q_{ii} comes from the third term of q_{ii} (i.e., $\sum_{j=1}^k b_{ij} \cdot a_{ij,i}$). Recall that b_{ij} is the coefficient of $\rho_{W_{i_j} Y}$ in the expression for $\phi_i(Z_i, Y, \mathbf{W}_i)$. From property (iii) of lemma 1, b_{ij} is a linear function on the correlations $\rho_{Z_i W_{i_1}}, \dots, \rho_{Z_i W_{i_k}}$, with no constant term. Moreover, correlation $\rho_{Z_i W_{i_j}}$ can be expressed as a sum of terms corresponding to unblocked paths between Z_i and W_{i_j} . Thus, every term in b_{ij} has the term of an unblocked path between Z_i and some W_{i_l} as a factor. By lemma 4, we get that any such path must include either an edge that does not belong to any of p_1, \dots, p_n , or an edge which appears in some of p_{i+1}, \dots, p_n . As above,

in both cases τ and T^* must be distinct terms.

After eliminating all those terms from consideration, the remaining terms in the product of $\langle 1, \dots, n \rangle$ are given by the expression:

$$T^* \cdot \prod_{i=1}^n (1 + \hat{b}_{i_0})$$

Since \hat{b}_{i_0} is a polynomial on the correlations among variables W_{i_1}, \dots, W_{i_k} , with no constant term, it follows that T^* appears only once in this expression. \square

Proposition: Term T^* does not appear in the product of any permutation other than $\langle 1, \dots, n \rangle$.

Proof: Let π be a permutation different from $\langle 1, \dots, n \rangle$, and let τ be a term in the product of π .

Let i be such that, for all $l > i$, π selects the diagonal entry in the row l of Q . As before, for $l > i$, if the factor of τ corresponding to entry q_{il} does not come from the first term of q_{il} (i.e., $T(p_l)[1 + \hat{b}_{l_0}]$), then τ must be different from T^* . So, we assume that this is the case.

Assume that π does not select the diagonal entry q_{ii} of Q . Then, π must select some entry q_{il} , with $l < i$. Entry q_{il} can be written as:

$$q_{il} = b_{i_0} a_{i_0 l} + \sum_{j=1}^{k_i} b_{i_j} a_{i_j l}$$

Assume that the factor of τ corresponding to entry q_{il} comes from term $b_{i_0} \cdot a_{i_0 l}$. Recall that each term in $a_{i_0 l}$ corresponds to an unblocked path between Z_i and Y including edge $X_l \rightarrow Y$. Thus, in this case, lemma 5 implies that τ and T^* are distinct terms.

Now, assume that the factor of τ corresponding to entry q_{il} comes from term $\sum_{j=1}^k b_{i_j} a_{i_j l}$. Then, by the same argument as in the previous proof, terms τ and T^* are distinct. \square

Hence, term T^* is not cancelled out and the lemma holds. \square

7.4 Identification of $\lambda_1, \dots, \lambda_n$

Lemma 8 gives that $\det(Q)$ is a non-trivial polynomial on the parameters of the model. Thus, $\det(Q)$ only vanishes on the roots of this polynomial. However, [8] has shown that the set of roots of a polynomial has Lebesgue measure zero. Thus, system Φ has unique solution almost everywhere.

It just remains to show that we can estimate the entries of the matrix of coefficients of system Φ from data.

Let us examine again an entry q_{il} of matrix Q :

$$q_{il} = \sum_{j=0}^k b_{i_j} \cdot a_{i_j l}$$

From condition (iii) of lemma 1, the factors b_{i_j} in the expression above are polynomials on the correlations among the variables $Z_i, W_{i_1}, \dots, W_{i_k}$, and thus can be estimated from data.

Now, recall that $a_{i_0 l}$ is given by the sum of terms corresponding to each unblocked path between Z_i and Y including edge $X_l \rightarrow Y$. Precisely, for each term t in $a_{i_0 l}$, there is an unblocked path p between Z_i and Y including edge $X_l \rightarrow Y$, such that t is the product of the parameters of the edges along p , except for λ_l .

However, notice that for each unblocked path between Z_i and Y including edge $X_l \rightarrow Y$, we can obtain an unblocked path between Z_i and X_l , by removing edge $X_l \rightarrow Y$. On the other hand, for each unblocked path between Z_i and X_l we can obtain an unblocked path between Z_i and Y , by extending it with edge $X_l \rightarrow Y$.

Thus, factor $a_{i_0 l}$ is nothing else but $\rho_{Z_i X_l}$. It is easy to see that the same argument holds for $a_{i_j l}$ with $j > 0$. Thus, $a_{i_j l} = \rho_{W_{i_j} X_l}$, $j = 0, \dots, k$.

Hence, each entry of matrix Q can be estimated from data, and we can solve the system of equations Φ to obtain the parameters $\lambda_1, \dots, \lambda_n$.

8 Conclusion

In this paper, we presented a generalization of the method of Instrumental Variables. The main advantage of our method over traditional IV approaches, is that it is less sensitive to the set of conditional independences implied by the model. The method, however, does not solve the Identification problem. But, it illustrates a new approach to the problem which seems promising.

Appendix

Proof of Lemma 1: Functions $\phi(1, \dots, n)$ and $\psi(i_1, \dots, i_{n-1})$ are defined recursively. For $n = 3$,

$$\begin{cases} \phi^3(1, 2, 3) &= \rho_{12} - \rho_{13}\rho_{23} \\ \psi^2(i_1, i_2) &= \sqrt{(1 - \rho_{i_1, i_2}^2)} \end{cases}$$

For $n > 3$, we have

$$\left\{ \begin{array}{l} \phi^n(1, \dots, n) = (\psi^{n-2}(n, 3, \dots, n-1))^4 \\ \quad \cdot \phi^{n-1}(1, 2, 3, \dots, n-1) \\ \quad - (\psi^{n-2}(n, 3, \dots, n-1))^2 \\ \quad \cdot \phi^{n-1}(1, n, 3, \dots, n-1) \\ \quad \cdot \phi^{n-1}(2, n, 3, \dots, n-1) \\ \psi^{n-1}(i_1, \dots, i_{n-1}) = \left[(\psi^{n-2}(i_1, i_2, \dots, i_{n-2}) \right. \\ \quad \cdot \psi^{n-2}(i_{n-1}, i_2, \dots, i_{n-2}))^2 \\ \quad \left. - (\phi^{n-1}(i_1, i_{n-1}, i_2, \dots, i_{n-2}))^2 \right]^{\frac{1}{2}} \end{array} \right.$$

Using induction and the recursive definition of $\rho_{12.3\dots n}$, it is easy to check that:

$$\rho_{12.3\dots N} = \frac{\phi^N(1, 2, \dots, N)}{\psi^{N-1}(1, N, 3, \dots, N-1) \cdot \psi^{N-2}(N, 3, \dots, N-1)}$$

Now, we prove that functions ϕ^n and ψ^{n-1} as defined satisfy the properties (i) – (iv). This is clearly the case for $n = 3$. Now, assume that the properties are satisfied for all $n < N$.

Property (i) follows from the definition of $\phi^N(1, \dots, N)$ and the assumption that it holds for $\phi^{N-1}(1, \dots, N-1)$.

Now, $\phi^{N-1}(1, \dots, N-1)$ is linear on the correlations $\rho_{12}, \dots, \rho_{N-1,2}$. Since $\phi^{N-1}(2, N, 3, \dots, N-1)$ is equal to $\phi^{N-1}(N, 2, 3, \dots, N-1)$, it is linear on the correlations $\rho_{32}, \dots, \rho_{N,2}$. Thus, $\phi^N(1, \dots, N)$ is linear on $\rho_{12}, \rho_{32}, \dots, \rho_{N,2}$, with no constant term, and property (ii) holds.

Terms $(\psi^{N-2}(N, 3, \dots, N-1))^2$ and $\phi^{N-1}(1, N, 3, \dots, N-1)$ are polynomials on the correlations among the variables $1, 3, \dots, N$. Thus, the first part of property (iii) holds. For the second part, note that correlation ρ_{12} only appears in the first term of $\phi^N(1, \dots, N)$, and by the inductive hypothesis $(\psi^{N-2}(N, 3, \dots, N-1))^4$ has constant term equal to 1. Also, since $\phi^N(1, 2, 3, \dots, N) = \phi^N(2, 1, 3, \dots, N)$ and the later one is linear on the correlations $\rho_{12}, \rho_{13}, \dots, \rho_{1N}$, we must have that the coefficients of $\phi^N(1, 2, \dots, N)$ must be linear on these correlations. Hence, property (iv) holds.

Finally, for property (iv), we note that by the inductive hypothesis, the first term of $(\psi^{N-2}(N, 3, \dots, N-1))^2$ has constant term equal to 1, and the second term has no constant term. Thus, property (iv) holds. \square

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