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A New Characterization of Graphs Based on Interception Relations

Azaria Paz
TECHNION—IIT
HAIFA 32000, ISRAEL

Judea Pearl
UNIVERSITY OF CALIFORNIA
LOS ANGELES, CA 90024

Shmuel Ur
IBM RESEARCH AND DEVELOPMENT CENTER
HAIFA, ISRAEL

ABSTRACT

While graphs are normally defined in terms of the 2-place relation of adjacency, we take the 3-place relation of interception as the basic primitive of their definition. The paper views graphs as an economical scheme for encoding interception relations, and establishes an axiomatic characterization of relations that lend themselves to representation in terms of graph interception, thus providing a new characterization of graphs.

1. INTRODUCTION

One of the main reasons that graphs offer useful representations for a wide variety of phenomena is that they display vividly the associations that exist among objects in the domain distinguishing direct from indirect associations. Traditionally, graph theory takes the notion of adjacency (or neighborhood) as a basic primitive, on the basis of which more elaborate notions, such as connectivity and interception, are defined and analyzed. In certain applications, the property of adjacency cannot be measured directly nor can it be defined uniquely in terms of the measured properties. Instead, adjacency can only be postulated as a convenient means for explaining associational phenomena resembling

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connectivity and interception, phenomena for which the distinction between direct and indirect association has clear operational definition in the domain.

A typical example is the notion of dependence and conditional dependence in probability theory. Given a probability function P on a collection of variables or events, it is straightforward to determine whether a pair of variables X and Y are dependent or independent, and whether X and Y are conditionally independent given a third variable Z. Yet P does not dictate which pairs of variables are considered adjacent. It is not even clear whether the notion of adjacency, hence graph theory, would be helpful in analyzing properties of conditional independence. While it is true that conditional independence bears similarity to interception in graphs, the similarity may not be complete, and it is not easy to determine what properties of conditional independence are mirrored by graph interception.

This paper takes the notion of interception as a basic primitive, and establishes necessary and sufficient conditions under which a relation of indirect associations can be faithfully represented by graph theoretical interception. When no faithful mapping exists, we then establish sufficient conditions for finding a unique best one-sided approximate representation in graphs. Thus, the paper lays a logical basis for topics such as Markov random fields [2], graphical models in statistics [3], and information relevance in artificial intelligence [4], where graphs are used primarily as a language for encoding complex patterns of mediated associations.

The authors of this paper are not aware of any other paper in the graph theoretic literature taking a similar approach. While the problem of graph separation was investigated by many authors (see, e.g. [5] or [6]), the main goal of those investigations was to find conditions under which a graph can be separated into subgraphs by removing a set of vertices, in the context of sometimes enabling "divide and conquer" algorithms and their recursive analysis.

2. DEFINITIONS AND NOTATIONS

Let V be a finite set $V = \{v_1, \ldots, v_n\}$, and let X, Z, Y denote finite subsets of V. We consider ternary relations over V as sets of triplets of the form (X, Z, Y). In the sequel we shall assume, unless otherwise specified, that all the relations R considered have the following properties:

(i) If $(X, Z, Y) \in R$ (notation: $R(X, Z, Y)$) then X, Z, Y are pairwise disjoint sets.
(ii) $R(X, Z, \emptyset)$ and $R(\emptyset, Z, X)$ for all disjoint X, Z, where $\emptyset$ denotes the empty set.

The elements of R will be called *R-triplets,* each triplet conveying the general notion of Z intercepting or mediating an indirect interaction between X and Y. R-triplets with $X = \emptyset$ or $Y = \emptyset$ or both will be called trivial triplets.

Undirected graphs (UG's) will be denoted by $G = (V, E)$, where V is the vertex-set and E is the edge-set of G. The graphs considered in this paper will be assumed to be simple and with no loops (i.e., if $(a, b) \in E$, then $a \neq b$).

Definition 1. Let $G = (V, E)$ be a graph. The relation R_G over V, *induced* by G, is defined as follows. $(X, Z, Y) \in R_G$ iff either X is disconnected from Y in G or Z is a cutset between X and Y in G. In addition, all trivial triplets are in R_G. ∎

Notice that in the above definition Z is not required to be a minimal cutset between X and Y. For any set of vertices $X \subseteq V, X$ is considered disconnected from $\emptyset$ by definition.

Definition 2. Let t be an R-triplet over V and $G = (V, E)$ be a given graph. t is *represented* (or *holds*) in G if an only if $t \in R_G$. ∎

Definition 3. Let $G = (V, E)$ be a graph and let R be a relation over V. G is an *I-map* of R if $R_G \subseteq R$. G is a *D-map* of R if $R \subseteq R_G$. If $R = R_G$, then R is represented by G. R is called graphical if $R = R_G$ for some graph G. ∎

Remark. The I-map and D-map definitions are borrowed from the applications to probabilistic distribution representations by graphs, where "I" stands for "Independency" and "D" stands for "Dependency" [4].

3. PROPERTIES OF R_G

Let G be a graph and let R be the relation induced by G. The subscript G will be omitted from R_G in the sequel and when understood.

Lemma 1. The relation R induced by G has the following properties:

(1) $R(X, Z, Y) \rightarrow R(Y, Z, X)$: Symmetry.
(2) $R(X, Z, Y \cup W) \rightarrow R(X, Z, Y) \wedge R(X, Z, W)$: Decomposition.
(3) $R(X, Z, Y) \rightarrow R(X, Z \cup W, Y)$ for all W disjoint from $X \cup Y$: Strong Union.
(4) $R(X, Z \cup W, Y) \wedge R(X, Z \cup Y, W) \rightarrow R(X, Z, Y \cup W)$: Intersection.
(5) $R(X, Z, Y) \rightarrow R(X, Z, \{a\}) \vee R(\{a\}, Z, Y)$ for any $a \in V, a \notin \{X \cup Z \cup Y\}$: Transitivity.

The above properties will be called the Graph Axioms in the sequel.

Proof. The proof of the first three properties is trivial and left to the reader.

Proof of the intersection property: Assume that the left-hand side of (4) holds for $R(= R_G)$. If X is disconnected in G from both Y and W then the right-hand side of (4) holds by definition.

If X is disconnected from Y, say, but is not disconnected from W, then $R(X, Z \cup Y, W)$ on the left-hand side of (4) implies that all the paths between X and W intersect Z. Thus $R(X, Z, Y \cup W)$. The case where X is disconnected from W only is similar.

The remaining case is the case where X is connected to both Y and W. Assume that this is the case and that the right-hand side of (4) does not hold. From $\neg R(X, Z, Y \cup W)$ we infer that there is a path from X to Y not intercepted by Z or there is a path from X to W not intercepted by Z. Assume the former w.l.o.g.. From $R(X, Z \cup W, Y)$ we infer that the path from X to Y not intercepted by Z must be intercepted by W. We conclude that there is a path from X to W, not intercepted by $Z \cup Y$ contrary to $R(X, Z \cup Y, W)$. This contradiction completes the proof.

Proof of Transitivity. If $R(X, Z, Y)$ and $a \notin \{X \cup Z \cup Y\}$, then either Z disconnects X from a or Z disconnects Y from a, since otherwise X is connected to Y via a path through a not intersecting Z. Thus the right-hand side of (5) must hold. ∎

Definition 4. Let R be a relation considered as a set of triplets. We shall say that R is closed under the graph axioms, or any other set of axioms, if whenever R satisfies the left-hand side of an axiom in the set it also satisfies its right-hand side, where the statement "$R(X, Z, Y)$" is understood as "the triplet (X, Z, Y) is in R." ∎

Corollary. If R is a graphical relation then R is closed under the graph axioms.

Lemma 2. Let R be any relation closed under the graph axioms. Then R also satisfies the following properties:

(6) $R(X, Z, Y) \wedge R(X, Z, W) \rightarrow R(X, Z, Y \cup W)$ where X, Z and $\{Y \cup W\}$ are pairwise disjoint sets.
(7) $R(X, Z, Y) \leftrightarrow (\forall a \in X)(\forall b \in Y) R(a, Z, b)$

Proof. Set $Y' = Y/W$. From $R(X, Z, Y)$ we get, by axiom (2), $R(X, Z, Y')$, and from $R(X, Z, Y') \wedge R(X, Z, W)$ we get, by axiom (3), $R(X, Z \cup W, Y') \wedge R(X, Z \cup Y', W)$, which imply, by axiom (4), $R(X, Z, Y' \cup W) = R(X, Z, Y \cup W)$. This proves property (6). Property (7) is a direct consequence of properties (1), (2), and (6). ∎

Remark. Let $V = \{a, b, c\}$. The relations $R_i, 1 \leq i \leq 5$, defined below, have the property that R_i is closed under all the graph axioms except axiom (i). It follows that the graph axioms are independent.

R_1: $\{(a, b, c)$ + all trivial triplets$\}$.
R_2: $\{(c, \emptyset, \{a, b\}), (c, a, b), (c, b, a)$ + symmetric triplets and all trivial triplets$\}$.
R_3: $\{(a, \emptyset, c), (b, \emptyset, c)$ + symmetric triplets and all trivial triplets$\}$.
R_4: $\{(a, b, c), (a, c, b)$ + symmetric triplets and all trivial triplets$\}$.
R_5: $\{(a, \emptyset, c), (a, b, c)$ + symmetric triplets and all trivial triplets$\}$.

4. THE GRAPH CHARACTERIZATION THEOREM

We can now prove our main theorem.

Theorem 1. Let R be a ternary relation over V. Iff R is closed under the Graph Axioms, then R is graphical.

Proof. The "easy" direction, namely "only if," follows from Corollary 1. To prove the "if" direction: given a relation R over V, construct the graph $G_0 = (V, E_0)$ such that for every pair $a, b \in V, a \neq b; (a, b) \in E_0$ iff $(a, V/\{a, b\}, b) \notin R$. (We use here and will use in the sequel the notation "a" for "$\{a\}$," "b" for "$\{b\}$," etc., where a, b, etc., are vertices.)

We split the proof into two parts.

First we prove that if R is closed under the axioms (1), (2), and (4), then the graph G_0, defined above, is an I-map of R. In the second part of the proof we will show that if R is closed under all the graph axioms, then G_0 is a D-map of R. Thus both parts together show that G_0 is a perfect map of R. Notice that G_0 is a uniquely defined undirected graph, if R is closed under symmetry.

Proof of* I*-mapness. Assume that R is closed under the axioms (1), (2), and (4). We show, by finite descending induction on the size of the middle set $|Z|$, that $R_{G_0} \subseteq R$ ($|S|$ denotes the number of elements in the set S).

Basis. If $t = (X, Z, Y)$ with $|Z| \in \{n, n-1\}$, then t is trivial and $t \in R$ (see (ii) in Section 2) and $t \in R_G$ (see definition 1). If $t = (a, V/\{a,b\}, b), |V/\{a,b\}| = n-2$, then t is represented in G_0 iff (a, b) is not an edge of G_0, iff $t \in R$, by the construction of G_0.

Step. Assume that all $t = (X, Z, Y) \in R_{G_0}$ with $|Z| = k$, for some $k (\leq n-2)$, are in R, and let $t' = (X', Z', Y') \in R_{G_0}$ be a triplet such that $|Z'| = k - 1 (< n-2)$. To show that $t \in R$, we distinguish between two subcases.

Subcase 1. $X' \cup Y' \cup Z' = V$. From $|Z'| = k - 1(< n-2)$ we infer that either $|X'| \geq 2$ or $|Y'| \geq 2$, and we may assume w.l.o.g. that $|Y'| \geq 2$ with $Y' = Y'' \cup c$, where c is a vertex not in Y''. Then $R_{G_0}(X', Z', Y'' \cup c) \rightarrow R_{G_0}(X', Z', c) \wedge R_{G_0}(X', Z', Y'')$ by decomposition (which holds for graph relations). By strong union, we get from the above that $R_{G_0}(X', Z' \cup Y'', c) \wedge R_{G_0}(X', Z' \cup c, Y'')$. By the induction hypothesis we get $R(X', Z' \cup Y'', c) \wedge R(X', Z' \cup c, Y'')$ since $|Z' \cup Y''|, |Z' \cup c| \geq k$. By intersection, which holds for R, we get $R(X', Z', Y'' \cup c) = R(X', Z', Y')$ as required.

Subcase 2. $X' \cup Y' \cup Z' \subsetneq V$. Let c be a vertex $c \notin X' \cup Y' \cup Z'$. From $R_{G_0}(X', Z', Y')$ we get, by transitivity, that $R_{G_0}(c, Z', Y') \vee R_{G_0}(X', Z', c)$. By strong union, we get from the above that $R_{G_0}(X', Z' \cup c, Y')$ and $R_{G_0}(c, Z' \cup X', Y') \vee R_{G_0}(X', Z' \cup Y, c)$. By induction, since now the size of the middle sets is at least k, we get

$$R(X', Z' \cup c, Y') \wedge [R(c, Z' \cup X', Y') \vee R(X', Z' \cup Y', c)].$$

By intersection and symmetry, which hold for R, we get from the above that

$$R(c \cup X', Z', Y') \vee R(X', Z', Y' \cup c).$$

Finally, by decomposition and symmetry, which hold for R we get $R(X', Z', Y')$ as required. We have thus shown that G_0 is an I-map of R.

Proof of* D*-mapness. Based on Lemma 2 it is enough to prove D-mapness (i.e., that $R(X, Z, Y)$ implies $R_{G_0}(X, Z, Y)$) for triplets of the form (a, Z, b) where a and b are single vertices. The proof is again by descending induction on the size of $|Z|$.

Basis. The proof for the case where $|Z| \in \{n, n-1\}$ is the same as in the previous part. For $|Z| = n-2$ we know that $t = (a, V/\{a,b\}, b) \in R_{G_0}$, iff (a, b) is not an edge of G_0, iff $t \in R$, by the construction of G_0.

Step. Assume that all $t = (a, Z, b) \in R$ with $|Z| = k$, for some $k(\leq n-2)$, are in R_{G_0}. Let $t' = (a', Z', b') \in R$ be a triplet such that $|Z'| = k - 1(< n-2)$. Then there is some $c \notin Z' \cup \{a', b'\}$. From $t' \in R$ we get by transitivity that $R(c, Z', b') \vee R(a', Z', c)$, and from this and t' we get by strong union that

$$R(a', Z' \cup c, b') \wedge [R(c, Z' \cup a', b') \vee R(a', Z' \cup b', c)].$$

By induction (since the middle sets have now size $\geq k$) we get

$$R_{G_0}(a', Z' \cup c, b') \wedge [R_{G_0}(c, Z' \cup a', b') \vee R_{G_0}(a', Z' \cup b', c)].$$

By intersection and symmetry we get $R_{G_0}(c \cup a', Z', b') \vee R_{G_0}(a', Z', c \cup b')$.

Finally by decomposition and symmetry we get from the above that $R_{G_0}(a', Z', b')$ holds, as required. This completes the proof of the D-mapness and of the "if" part of the theorem. ∎

Not all the graph axioms are needed to guarantee the existence of a unique minimal I-map G_0. In Corollary 2 below we give a weaker set of axioms, which is sufficient to provide this guarantee.

Corollary. Let R be a relation over V, closed under the axioms (1), (2), and (4). Then a unique graph $G = (V, E)$ can be constructed such that $R_G \subseteq R$ (i.e., G is an I-map of R) and such that G is edge minimal (i.e., if an edge is removed from G then its I-mapness is violated).

Proof. The first part of the corollary follows from the first part of the proof of Theorem 1, showing that G_0 is an I-map of R, under the condition of the corollary. To see that G_0 is edge minimal, observe that if any edge (a, b) is removed from G_0; then (at least) the triplet $t = (a, V/\{a, b\}, b)$ is added to R_{G_0}, but this triplet is not in R since $t \in R$ implies that $(a, b) \notin G_0$, contrary to our assumption, by the definition of G_0. The same argument shows that if $G = (V, E)$ is any I-map of R, then $E_0 \subseteq E$. Conversely, adding edges to G_0 can only remove (never add) triplets to the induced relation, so that the resultant graph is again an I-map of R. Thus G_0 is the intersection of all I-maps of R, hence the unique minimal one. ∎

Notice that an alternative wording of Corollary 2 is

Let R be a relation over V, closed under axioms (1), (2), and (4). Then R contains a unique triplet-maximal graphical relation.

5. EXTENSIONS

Definition 5. Let Σ be a set of triplets over a set V. A relation R, over V, is an extension of Σ (notation R_Σ) if it satisfies the following conditions:

1. $\Sigma \subseteq R$.
2. R is graphical.

R is a minimal extension of Σ if no proper subset of R is an extension of Σ. R is a minimum extension of Σ if any other extension R' of Σ satisfies $|R'| \geq |R|$. ∎

Example 1. Let $\Sigma = \{(a, c, b), (a, d, b)\}$ over $\{a, b, c, d\}$. The relations shown below are minimal extensions of Σ and both are at the same time minimum extensions too.

$$\begin{aligned} R_1 &= \{(a, \{b, d\}, c), (a, \{c, d\}, b), (b, \{a, c\}, d), (a, d, \{b, c\}), (b, c, \{a, d\}), (a, c, b), (a, d, c), \\ &\quad (a, d, b), (b, c, d) + \text{symmetric triplets} + \text{trivial triplets}\} \\ R_2 &= \{(a, \{b, c\}, d), (a, \{c, d\}, b), (b, \{a, d\}, c), (a, c, \{b, d\}), (b, d, \{a, c\}), (a, c, b), (a, c, d), \\ &\quad (a, d, b), (b, d, c) + \text{symmetric triplets} + \text{trivial triplets}\}. \end{aligned}$$

There are additional extensions that are minimal but not minimum, e.g., the graphical relation induced by the graph consisting of the isolated vertex a and a triangle with vertices

b, c, d. The extension including all the possible triplets over V is neither minimal nor minimum.

An algorithm is shown below that provides a minimal extension of a given set Σ, with time complexity that is polynomial in the size of Σ. Finding an extension that is minimum, in polynomial time, is an open problem.

Another interesting open problem is to characterize the sets Σ for which it is true that all minimum extensions yield isomorphic graphs.

AN ALGORITHM FOR FINDING A MINIMAL EXTENSION OF A GIVEN SET OF TRIPLETS Σ OVER A SET V

1. Start with the complete graph over V and remove all edges (a, b) such that $a \in X$ and $b \in Y$ for some $(X, Z, Y) \in \Sigma$. Denote the resulting graph by G_Σ.

2. If Σ (i.e., all the triplets in Σ) is represented in G_Σ, then return G_Σ.

3. Let $\sigma = (X, Z, Y)$ be the first triplet in Σ not represented in G_Σ. This implies that there are vertices a, b, c in V such that $a \in X, b \in Y$, and $c \notin X \cup Y \cup Z$, and such that there is a path from a to b in G_Σ passing through c and not passing through Z (this follows from the fact that as a result of Step 1, in G_Σ no vertex in X is adjacent to a vertex in Y). Choose c as above to be a vertex with at least one neighbor in X (i.e., c is chosen to be the first vertex outside of $X \cup Y \cup Z$ on a path between a and b outside of Z). Reset G_Σ by removing from it all edges connecting c to a vertex in X. Go to 2.

End of algorithm.

The number of iterations of the algorithm is $O(n^2)$, since at every iteration at least one edge is removed from G_Σ. The number of operations at every iteration is $O(|\Sigma|n^4)$ (since for every element (X, Z, Y) in Σ we must check whether a path exists between a vertex in X and a vertex in Y when the vertices in Z are removed. Thus an $O(n^2)$ search is required for every pair $x \in X$ and $y \in Y$). The algorithm is therefore polynomial in the size of Σ.

The graph G_Σ output by the algorithm defines the relation R_{G_Σ}, which includes Σ as a subset and is closed under the graph axioms. Thus R_{G_Σ} is an extension of Σ. That the extension is minimal can be shown as follows: Every extension of Σ is a relation closed under the graph axioms. By (the characterization) Theorem 1, every relation closed under the graph axioms is graphical. The algorithm directly constructs a graph representing such an extension and the edges removed at Steps 1 and 3 are a minimal set of edges whose removal is necessary in order to enable the representation of Σ in the graph.

Remark. It is an open problem to prove that the algorithm given here is optimal or to provide a faster algorithm instead.

6. SOUNDNESS AND COMPLETENESS

Denote by A the set of graph axioms and by $\mathcal{G}$ the family of simple undirected graphs with no loops.

In the definitions and results below, a common set of vertices V of size n is assumed and all the triplets considered are triplets over this set.

Definition 6. Let Σ be a set of triplets and σ a single triplet. Σ A-derives σ (notation: $\Sigma \vdash_A \sigma$) iff σ is an element of every extension of Σ. ∎

The relation of A-derivation to the usual concept of deductive derivation will be given in Definition 9.

Definition 7. Let Σ be a set of triplets and σ a single triplet. Σ $\mathcal{G}$-implies σ (notation: $\Sigma \models_{\mathcal{G}} \sigma$) iff for any graph $G \in \mathcal{G}, \Sigma \subset R_G$ implies that $\sigma \in R_G$. ∎

Definition 8. The set A of axioms is sound for $\mathcal{G}$ if $\Sigma \vdash_A \sigma$ implies that $\Sigma \models_{\mathcal{G}} \sigma$. The set of axioms is complete for $\mathcal{G}$ if $\Sigma \models_{\mathcal{G}} \sigma$ implies $\Sigma \vdash_A \sigma$ for any set of triplets Σ and single triplet σ. ∎

Theorem 2. The graph axioms A are sound and complete for $\mathcal{G}$.

Proof of Soundness. Let G be any graph in $\mathcal{G}$, assume that Σ is represented in G, and assume that $\Sigma \vdash_A \sigma$. R_G is an extension of Σ and therefore, from $\Sigma \vdash_A \sigma$, we get that σ is represented in R_G as required.

Proof of Completeness. Given Σ and σ, let R_Σ be an extension of Σ. By Theorem 1, there is a graph G that is a perfect map of R_Σ. Thus $\Sigma \subset R_G = R_\Sigma$. From $\Sigma \models_{\mathcal{G}} \sigma$ we get that $\sigma \in R_G$. Thus $\sigma \in R_\Sigma = R_G$. ∎

The concept of an A-derivation, as defined in Definition 6, depends on the concept of an extension. The usual (and stronger) concept of A-derivation is defined below.

Definition 9. Let Σ be a set of triplets and σ a single triplet. Σ strongly A-derives σ (notation: $\Sigma \| -_A \sigma$) if σ can be derived from Σ by a deductive chain of formulas $f_1, f_2, \ldots, f_k$ such that $f_k = \sigma$ and every $f_i, i < k$, is a boolean formula of triplets such that either $f_i \in \Sigma$ or f_i is derived from previous f_j's in the chain as a derivation of the predicate calculus extended by the A-axioms. ∎

Example 1. Let $\Sigma = \{(3,2,\{1,4\}),(1,2,\{3,4\}),(1,4,3)\}$, and let $\sigma = (1,\emptyset,3)$, over $V = \{1,2,3,4\}$. Below is a derivation chain for σ.

f_1: $(1,4,3) \in \Sigma$.
f_2: $(2,4,3) \vee (1,4,2)$ by transitivity.
f_3: $(2,\{1,4\},3) \vee (1,\{3,4\},2)$ from f_2 by strong union and predicate calculus.
f_4: $(\{1,4\},2,3)$ from Σ by symmetry.
f_5: $(1,2,\{3,4\}) \in \Sigma$.
f_6: $(\{1,2,4\},\emptyset,3) \vee (1,\emptyset,\{2,3,4\})$ from f_3, f_4, and f_5 by symmetry, intersection, and predicate calculus.
f_7: $(1,\emptyset,3) \vee (1,\emptyset,3)$ from f_6 by symmetry decomposition and predicate calculus.
f_8: $(1,\emptyset,3)$ from f_7 by predicate calculus.

It is easy to see that strong A-derivation implies A-derivation: Let $f_1,\ldots,f_k$ be a strong derivation of σ from Σ, and let R_Σ be an extension of Σ. Then $f_i, 1 \le i \le k$, holds in R_Σ, since $\Sigma \subset R_\Sigma$ and R_Σ is closed under A. It follows that $\sigma = f_k \in R_\Sigma$ so that Σ A-derives σ. We have thus proved

Lemma 3. For any set of triplets Σ and single triplet $\sigma, \Sigma \models_A \sigma$ implies that $\Sigma \vdash_A \sigma$.

The question whether $\Sigma \vdash_A \sigma$ implies $\Sigma \models_A \sigma$ is open. A positive answer to this question would assert (based on the fact that every extension of Σ has a perfect graph representation) that every valid cut-set graph property of the form "For any graph G, if Σ holds in R_G then σ holds in R_G" can be proved in the predicate calculus when extended by the graph axioms. Consider, e.g., again the example above. The example can be extended to the following: Let $G(V, E)$ be a graph and let X, Y, Z, W be a partition of V such that Y is a connected set of vertices. Notice that a generalized axiom of transitivity holds in a graphical relation R_G, when the vertex a in axiom (5) is replaced by a set of vertices Y, only if Y is a connected set of vertices. If $(Z, Y, X \cup W), (X, Y, Z \cup W)$, and (X, W, Z) hold in R_G, then G has at least 2 components with X in one component and Z in the other (which is equivalent to $(X, \emptyset, Z) \in R_G$). The example shows that this particular property can be proved in the predicate calculus when extended by the graph axioms. The question whether *every* valid property of graph separation can be decided by these means depends on whether $\vdash$ implies $\Vdash$.

7. NP-COMPLETENESS OF WEAK INDEPENDENCE

Let Σ be a set of triplets and σ a triplet over V. We shall say that σ is weakly independent of Σ if $\Sigma \vdash_A \sigma$ does not hold and σ is strongly independent of Σ if $\Sigma \Vdash_A \sigma$ does not hold. It follows from Lemma 3 that weak independence implies strong independence. It follows from Theorem 2 that σ is weakly independent of Σ (notation: $\Sigma \nvdash_A \sigma$) if and only if there exists a graph G such that Σ is represented in G and σ is not represented in G. We will show now that the problem of ascertaining whether a graph G as above (representing Σ and not representing σ) exists for any given Σ and σ, is NP-complete.

That the problem is in NP is trivial since for any given Σ and σ we can guess, in polynomial time, a graph, and then check, in polynomial time, whether it has the required property. To show NP-completeness we will present a polynomial reduction from the Hamiltonian problem, a well-known NP-complete problem (see [1]), to the weak independence problem. We first set the definitions in standard form:

Hamiltonian.

Input. A graph $G(V, E)$ and a pair of vertices $a, b \in V$.

Problem. Does there exist a path in G from a to b passing through every vertex in V exactly once?

Independence.

Input. A set of triplets Σ over V and a triplet σ over V.

Problem. Does there exist a graph $G = (V, E)$ such that Σ is represented in G and σ is not represented in G?

Theorem 3. The independence problem is NP-complete.

Proof. We have already shown that Independence is in NP. Consider now the following reduction. Given $G = (V, E)$ and $a, b \in V$, an input for the Hamiltonian problem set:

$$\Sigma_1 = \{(u, V \setminus \{u, v\}, v)\colon (u, v) \notin E\}$$

$$\begin{aligned}\Sigma_2 &= \{(a, v, b)\colon v \in V/\{a, b\}\}\\ \Sigma &= \Sigma_1 \cup \Sigma_2\\ \sigma &= (a, \emptyset, b).\end{aligned}$$

It is clear that Σ and σ can be set in polynomial time. To complete the proof of the theorem, it suffices to prove the following claim: The Hamiltonian problem with input G and a, b has a solution if and only if the independence problem has a solution with input Σ and σ.

To prove this claim we notice first that every graph G' that satisfies Σ_1 is a subgraph of G, by the definition of Σ_1. If in addition G' does not satisfy σ then there must be a path in G' between a and b. Finally, if G' satisfies Σ_2, then the path in G' between a and b must be intercepted by every vertex in V exactly once (every vertex in V must disconnect between a and b, as required by Σ_2, and the path cannot have a loop since otherwise the vertices on the loop will not disconnect a from b). Thus, if G' satisfies Σ and does not satisfy σ, then it has a Hamiltonian between a and b and this Hamiltonian exists in G, since G' is a subgraph of G. On the other hand, if G has a Hamiltonian G' between a and b, then G' is a subgraph of G satisfying Σ and not satisfying σ, as is easy to see. ∎

8. NEIGHBORHOODS

Given a relation R over a set V, it is only when R is closed under the graph axioms that the proof of Theorem 1 provides a method for constructing a graph G_0 such that $R_{G_0} = R$. If R is closed under the symmetry, decomposition, and intersection axioms, then G_0 is only assured to be an edge minimal I-map of R. An intermediate situation will be considered in this section. We will show that if R is closed under the axioms of symmetry, decomposition, intersection, and weak union—an axiom to be defined below—then the approximation provided by G_0 for R is not only an edge minimal I-map of R, but is stronger in the sense that it encodes and unifies two diverse notions of neighborhood in R.

Since each (X, Z, Y) triplet in R conveys the informal notion of broken interaction (Z breaks the interaction between X and Y), there are two natural ways of defining neighborhood. One is to proclaim a pair of elements a and b neighbors iff their interaction cannot be broken even by all the other elements in U—namely, $(a, U/\{a, b\}, b) \notin R$. Alternatively, we may wish to define the neighbors of a as a minimal set of elements needed to break the interaction between a and all other elements of U. We will show that under certain conditions these two notions of neighborhood will become identical, and will coincide with ordinary adjacency in G_0.

The following property, for a relation R over a set V, will be called the axiom of *weak union:*

$$R(X, Z, Y \cup W) \to R(X, Z \cup Y, W) \wedge R(X, Z \cup W, Y) \tag{8}$$

where the sets X, Y, Z, W are assumed to be pairwise disjoint.

Notice that a relation that satisfies strong union and decomposition also satisfies weak union; since we can remove first, by decomposition, W or Y from the right-hand side of $R(X, Z, Y \cup W)$ and then reinsert the removed set in the middle, by strong union.

Let R be a relation over V, let a be a vertex in V, and let G_0 be the graph defined in the proof of Theorem 1 for the given R. Given that R is closed under symmetry, G_0 is a uniquely defined undirected graph. Define the set $S(a)$ as below:

$$S(a) = \{b\colon\ R(a, V/\{a,b\}, b), b \in V\}.$$

By the definition of G_0, the set of $S(a)$ is the set of vertices in G_0 that are not connected by an edge to a in G_0. It follows that $V/\{S(a) \cup a\}$ is the set of neighbors of a in G_0, the set of vertices in G_0 connected by an edge to a in G_0. Denote this set by $N(a)$, i.e., $N(a) = V/\{S(a) \cup a\}$. For the given R, V, and $a \in V$ we define also the following sets:

$$\begin{aligned} \varphi(a) &= \{X\colon\ R(a, X, V/\{X \cup a\})\} \\ BL(a) &= \text{the set in } \varphi(a) \text{ whose size } (= \text{number of elements}) \text{ is minimal.} \end{aligned}$$

$BL(a)$ stands for the "blanket of a" since it is a set of minimal size shielding a from the rest of the vertices.

Theorem 4. Let R be a relation closed under the axioms of symmetry, decomposition, weak union, and intersection. Then $BL(a)$ is uniquely defined and $BL(a) = N(a)$.

Proof. We will show that for any set $X \subseteq V$ such that $a \notin X, X \in \varphi(a)$ if and only if $X \supseteq N(a)$. This implies the claim of the theorem since it shows that all the sets in $\varphi(a)$ are supersets of $N(a)$ and it shows that $N(a)$ itself is a set in $\varphi(a)$ so the set of minimal size in $\varphi(a)$ must equal to $N(a)$.

Proof of the "if" Part. Assume $X \supseteq N(a)$. Then $X = V/\{Y \cup a\}$, where $Y \subseteq S(a)$. We prove, by induction on the size of Y, that $X \in \varphi(a)$.

Basis. The claim is trivial for $Y = \emptyset$ since for this case $X = V/\{a\}$ and $R(a, V/\{a\}, \emptyset)$, which holds by assumption, implies that $X \in \varphi(a)$. If $|y| = 1$ or $Y = y \in S(a)$, then, by the definition of $S(a), R(a, V/\{a, y\}, y) = R(a, X, y)$ by our assumption on X. Thus $X \in \varphi(a)$.

Step. Assume that the claim is true for $Y_1, |Y_1| \geq 1$ and let $Y = \{Y_1 \cup y\} \subseteq S(a)$ where y is a singleton not in Y_i.

We can assume, by induction, that $X_1 = V/\{Y_1 \cup a\}$ and $X_2 = V/\{y \cup a\}$ are in $\varphi(a)$ so that

$$R(a, V/\{Y_1 \cup a\}, Y_1) \wedge R(a, V/\{y \cup a\}, y).$$

By intersection we get from the above that

$$R(a, V/\{Y_1 \cup a \cup y\}, Y_1 \cup y\} = R(a, V/\{Y \cup a\}, Y)$$

or $X = V/\{Y \cup a\} \in \varphi(a)$ as required.

Proof of the "only if" Part. Assume that $X \in \varphi(a)$. Then $R(a, X, V/\{X \cup a\})$. If X is not a superset of $N(a)$, then we can set $N(a) = X_1 \cup X_2, X = X_1 \cup X_3$, and $X_2 \neq \emptyset; X_1, X_2, X_3$ pairwise disjoint.

Now $R(a, X, V/\{X \cup a\}) = R(a, X_3 \cup X_1, V/\{X \cup a\})$ is given. Also $R(a, N(a), V/\{N(a) \cup a\}) = R(a, X_1 \cup X_2, V/\{N(a) \cup a\})$ since, by the first part of the proof $N(a)$, is in $\varphi(a)$.

All the elements of V are included in each of the above triplets with X_2 a subset of the right-hand side of the first triplet and X_3 a subset of the right-hand side of the second. We can therefore move, by weak union, all elements in the right-hand side not in X_2 to the middle part in the first triplet and all the elements in the right-hand side not in X_3 to the middle part of the second triplet, resulting in

$$R(a, V/\{X_2 \cup a\}, X_2) \wedge R(a, V/\{X_3 \cup a\}, X_3).$$

By intersection we get

$$R(a, V/\{X_2 \cup X_3 \cup a\}, X_2 \cup X_3).$$

But we assumed that $X_2 \neq \emptyset$. Let b be a vertex in X_2. Using weak union again, we can move all the elements except b from the right-hand side of the above triplet to the middle, resulting in

$$R(a, V/\{a, b\}, b).$$

On the other hand, $b \in X_2 \subseteq N(a)$ and $N(a)$ is disjoint from $S(a)$, implying that $b \notin S(a)$ which, by the definition of $S(a)$, implies that $\neg R(a, V/\{a, b\}, b)$, a contradiction. We must therefore conclude that $X \supseteq N(a)$. ∎

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