In J.G. Daunt, D.O. Edwards, F.J. Milford, and M. Yaqub (Eds.), Low Tembperature Physics LT9 {Proceedings of the IXth International Conference on Low Temperature Physics), NY: Plenum Press, 1965.

DISTINCTIVE PROPERTIES OF QUANTIZED VORTICES IN SUPERCONDUCTING FILMS*

J. Pearl

RCA Laboratories
Princeton, New Jersey

This work is an extension of Abrikosov's[1] model for the magnetic properties of thin superconducting films. We first derive the current distribution and the pair interaction forces for individual vortices, and from the results determine the macroscopic behavior of thin films in the presence of external magnetic fields. We restrict our discussion to ideal films, relying on the premise that vortices maintain the same constituent characteristics in the presence of film defects, save that their motion is impeded by pinning forces.

Consider the Ginzburg–Landau equations for an ideal superconducting film of thickness d and infinite extent, located in the x–y plane,

$$\left(i\nabla + 2\pi\frac{\mathbf{A}}{\phi_0}\right)^2\psi = \frac{1}{\xi^2}(\psi - |\psi|^2\psi) \tag{1}$$

$$\begin{aligned} -\operatorname{curl}\operatorname{curl}\mathbf{A} &= \frac{1}{\lambda^2}\left\{|\psi|^2\mathbf{A} + i\frac{\phi_0}{2\pi}\tfrac{1}{2}(\psi^*\nabla\psi - \psi\nabla\psi^*)\right\} \quad |z| \leqslant d/2 \\ &= 0 \qquad\qquad |z| > d/2 \end{aligned} \tag{2}$$

where ξ is the coherence length, $\phi_0 = ch/2e$ is the flux quantum, and ψ is the normalized order parameter. The first equation is identical to that treated by Abrikosov for bulk superconductors. The second equation, however, is three dimensional, as the geometry no longer possesses the z invariance. It is at this point that Abrikosov's solution is to be modified.

We follow Abrikosov and assume that the film carries an assembly of vortices sufficiently separated from each other so that each can be treated individually. For a single vortex located at the origin of the cylindrical coordinates (r, θ, z), we can use the axial symmetry and require ψ to be a single-valued function. This leads to the form

$$\psi = f(r)e^{in\theta} \tag{3}$$

where n is the flux quantum number. Since for $\lambda/\xi \gg 1$, f varies considerably only for distances smaller than ξ, one may assume f to be unity everywhere except in a narrow core of radius ξ. For $d/\lambda \ll 1$, the current density in the film is essentially uniform and may be approximated by an infinitesimally thin current sheet

$$\mathbf{J}(r, z) = \delta(z)\mathbf{K}(r) \tag{4}$$

* This work is part of a thesis to be submitted in partial fulfillment of the requirements for the Ph.D. degree in the Department of Electrical Engineering at the Polytechnic Institute of Brooklyn, N.Y.

With these simplifications, the second G–L equation assumes the form

$$-\operatorname{curl}\operatorname{curl}\mathbf{A} = -\frac{4\pi}{c}\delta(z)\mathbf{K}(r) = \frac{d}{\lambda^2}\delta(z)\left(\mathbf{A} - \frac{n\phi_0}{2\pi}\frac{1}{r}\hat{\theta}_0\right) \tag{5}$$

resulting in[2]

$$\begin{aligned} K(r) &= \frac{c}{4\pi}\left(\frac{d}{2\lambda^2}\right)^2\frac{\phi_0}{2}\left[S_1\left(\frac{rd}{2\lambda^2}\right) - N_1\left(\frac{rd}{2\lambda^2}\right) - \frac{2}{\pi}\right] \\ K(r) &= \frac{c}{4\pi}\frac{d}{2\lambda^2}\frac{n\phi_0}{\pi}\frac{1}{r} \qquad r \ll 2\lambda^2/d \\ K(r) &= \frac{c}{4\pi}\frac{n\phi_0}{\pi}\frac{1}{r^2} \qquad r \gg 2\lambda^2/d \end{aligned} \tag{6}$$

While for small values of r the current is identical with the Abrikosov solution for bulk superconductors, the behavior at far distances is drastically different; instead of an exponential cutoff, we now find a slow decay of currents following a $1/r^2$ law.

An insight into the reasons for these differences can be obtained by Fourier analyzing equation (5). We first notice that the first term on the right represents the regular London diamagnetic current, while the second term is equivalent to an externally applied current situated at $z = 0$. Thus, the current distribution in the film will be determined by the ability of the film to screen out the effect of external currents, j_e, applied very close to its surface. The vector potential for such a film will satisfy:

$$\nabla^2\mathbf{A} = -\frac{4\pi}{c}\mathbf{j}_t = \left(\frac{1}{\lambda^2}\mathbf{A} - \frac{4\pi}{c}\mathbf{j}_e\right)d\delta(z) \tag{7}$$

which has the Fourier transform

$$q^2\mathbf{A} = \frac{d}{2\pi}\int\left[\frac{4\pi}{c}\mathbf{j}_e(\mathbf{q}) - \frac{1}{\lambda^2}\mathbf{A}(\mathbf{q})\right]dq_z \tag{8}$$

resulting in the film permeability[2]

$$\mu_f(\mathbf{q}) = \frac{j_t(\mathbf{q})}{j_e(\mathbf{q})} = \frac{2q\lambda^2/d}{1 + 2q\lambda^2/d} \tag{9}$$

For bulk material we have the same differential equation (7) except for the absence of the terms $d\delta(z)$ and $\partial^2/\partial z^2$, giving

$$\mu_b(q) = \frac{\lambda^2 q^2}{1 + \lambda^2 q^2} \tag{10}$$

Figure 1 shows that thin films are more permeable than bulk material to low q components of fields (which are the ones responsible for the far field behavior). In films, distant regions are electromagnetically coupled through free space, while in bulk samples, regions are screened from each other by free electrons.

Knowing the current distribution around a single vortex, we are able to derive some of its basic properties, and compare thin film vortices to vortices in bulk superconductors (Table I). The excitation energies (U) have almost the same expression, but the pair interaction energies (U_{ij}) are different; for large distances we have a cutoff interaction in bulk material and a long range Coulomb interaction ($1/r_{ij}$) in

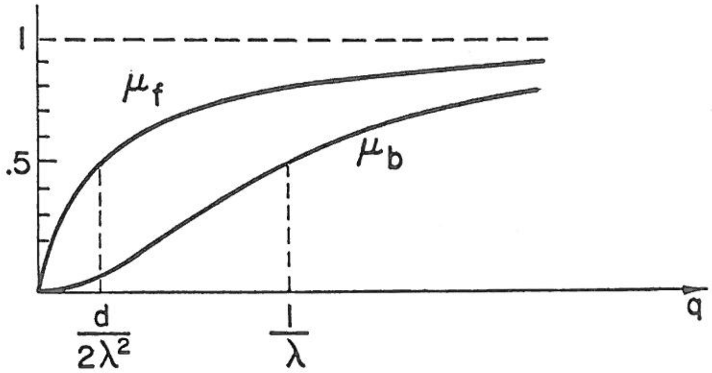


Fig. 1. Wave-number dependence of magnetic permeabilities for thin-film and bulk superconductors.

thin films. The magnetic moment associated with a single vortex in thin films is larger than the one in bulk superconductors by the factor R/d, R being the sample radius.

Consider now the effect of these properties on the macroscopic behavior of an assembly of loosely coupled vortices (no overlapping cores). The Ginzburg–Landau expression for the Gibbs' free energy for a superconductor in an external field H_0, supporting a total of N vortices, is given by

$$G(H_0, N) - F_{s0} = \frac{H_{cb}^2}{8\pi}\int(1 - |\psi|^2)^2\, dv + \frac{1}{2m}\int\left| - i\hbar\nabla\psi - \frac{e}{c}\mathbf{A}\psi\right|^2 dv + \frac{1}{8\pi}\int \mathbf{H}^2\, dv - \frac{1}{4\pi}\int \mathbf{H}_0\cdot\mathbf{H}\, dv \tag{11}$$

With our special form of ψ, and after being integrated by parts, this becomes

$$G(H_0, N) - F_{s0} \cong N\frac{H_{cb}^2}{8\pi}\pi\xi^2 d + \frac{1}{8\pi}\int[(\mathbf{H} - \mathbf{H}_0)^2 + 4\pi\Lambda\mathbf{j}^2]\, dv \tag{12}$$

$$\cong G(H_0, 0) + \sum_i^N U_i + \sum_{i<j}^N U_{ij} - H_0\sum_i^N M_i \tag{13}$$

Table I. Basic Properties of a Single Quantum Vortex in Bulk and Thin Film Superconductors ($n = 1$, $\lambda \gg \xi$)

	Bulk		Film	
K_s	$\frac{c}{4\pi}\frac{\phi_0 d}{2\pi\lambda^2}\frac{1}{r}$	$r \ll \lambda$	$\frac{c}{4\pi}\frac{\phi_0 d}{2\pi\lambda^2}\frac{1}{r}$	$r \ll \frac{2\lambda^2}{d}$
	$\frac{c}{4\pi}\frac{\phi_0 d}{2\lambda^2(2\pi\lambda)^{\frac{1}{2}}}\frac{e^{-r/\lambda}}{r^{\frac{1}{2}}}$	$r \gg \lambda$	$\frac{c}{4\pi}\frac{\phi_0}{\pi}\frac{1}{r^2}$	$r \gg \frac{2\lambda^2}{d}$
U	$\left(\frac{\phi_0}{4\pi\lambda}\right)^2 d\left(\ln\frac{\lambda}{\xi} + \ln 2 - \gamma\right) + \pi\, d\xi^2\frac{H_c^2}{8\pi}$		$\left(\frac{\phi_0}{4\pi\lambda}\right)^2 d\left(\ln\frac{\lambda}{\xi} + \ln\frac{4\lambda}{d} - \gamma\right) + \pi\, d\xi^2\frac{H_c^2}{8\pi}$	
U_{ij}	$\left(\frac{\phi_0}{2\pi\lambda}\right)^2 d\left(\ln\frac{\lambda}{r_{ij}} + \ln 2 - \gamma\right)$	$r_{ij} \ll \lambda$	$\left(\frac{\phi_0}{2\pi\lambda}\right)^2 d\left(\ln\frac{\lambda}{r_{ij}} + \ln\frac{4\lambda}{d} - \gamma\right)$	$r_{ij} \ll \frac{2\lambda^2}{d}$
	$\frac{\phi_0^2}{8\pi(2\pi)^{\frac{1}{2}}}\frac{d}{\lambda^2}\left(\frac{\lambda}{r_{ij}}\right)^{\frac{1}{2}} e^{-r_{ij}/\lambda}$	$r_{ij} \gg \lambda$	$\left(\frac{\phi_0}{2\pi}\right)^2\frac{1}{r_{ij}}$	$r_{ij} \gg \frac{2\lambda^2}{d}$
M	$\frac{\phi_0}{4\pi}d$		$\approx\frac{\phi_0}{4\pi}R$	

where U_i and M_i are the excitation energy and magnetic moment of the ith vortex. Equation (13) can be rewritten

$$G(H_0, N) - F_{s0} = G(H_0, 0) + N\overline{U} + f(N) - H_0 N\overline{M} \tag{14}$$

where $\overline{U}$ and $\overline{M}$ are the average energy and magnetic moment of a single vortex, and $f(N)$ is the total interaction energy. In equilibrium N is determined by the condition $\partial G/\partial N = 0$, which is satisfied when a virtual increase in the internal energy is balanced by a decrease in external pressure. H_{c1} is the field just sufficient to create a single vortex.

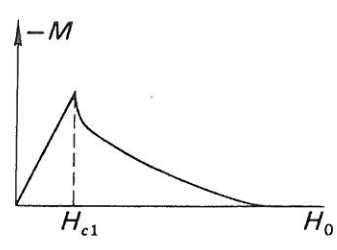
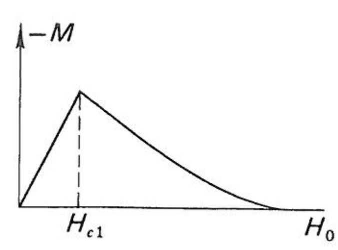
$$H_{c1} = \frac{\overline{U}}{\overline{M}} \tag{15}$$

The magnetization slope is given by

$$\frac{dM}{dH_0} = C + \overline{M}\frac{dN}{dH_0} = C + 2\overline{M}^2 \Big/ \frac{d^2 f}{dN^2} \tag{16}$$

A comparison of the reversible magnetization curves for bulk and thin film superconductors is given in Table II. The first critical field in thin films is lower than that of bulk by a factor d/R. (It is also lower than the one calculated[3] on the basis of current quenching at the film edges. This implies the existence of a surface barrier opposing the entry of vortices at the edge of the film.)

Table II. Reversible Magnetization Curves for Bulk and Thin Film Superconductors

	Bulk		Film
$\dfrac{H_{c1}}{H_{cb}} = \dfrac{\overline{U}}{\overline{M}H_{cb}}$	$\approx \dfrac{\xi}{\lambda}\left[\ln\dfrac{\lambda}{\xi} + 0(1)\right]$		$\approx \dfrac{d}{R}\dfrac{\xi}{\lambda}\left[\ln\dfrac{\lambda}{\xi} + 0(1)\right]$
$f(N) = \sum_{i>j} U_{ij}$	$\cong 0$	$r_{ij} > \lambda$	
	$\cong \dfrac{\phi_0^2 N^2}{R} 0\left(\dfrac{d}{R}\right)$	$\xi < r_{ij} < \lambda$	$\cong \dfrac{\phi_0^2 N^2}{R} 0(1)$
$\left.\dfrac{dM}{dH_0}\right\|_{H_{c1}}$	$\to \infty$		Finite $0(R^3)$
$\left.\dfrac{dM}{dH_0}\right\|_{H_0 > H_{c1}} = C + 2\overline{M}^2/f''(N)$	$0(R^2 d)$		$0(R^3)$
			

The interaction energy becomes proportional to N^2 whenever the expression ΣU_{ij} can be replaced by an integral, in which case the magnetization has a constant slope. In bulk superconductors this is valid only when the mean distance between vortices is small compared with their range of interaction. For thin films, however, the interaction range is infinite and $f(N)$ becomes proportional to N^2 even for small values of N. As a result while for bulk dM/dH_0 is infinite at the first critical field, dM/dH_0 for films is finite and is of order R^3.

An additional consequence of the long range fluxoid interaction is that the total free energy is extremely insensitive to the exact nature of the fluxoid lattice structure. As a result, the shear modulus vanishes, and the vortices do not assume a particular lattice order. On the other hand, the energy becomes very sensitive to deviations from the equilibrium density distribution, that is, the elastic modulus is high. In this respect the motion of fluxoids past pinning sites is analogous to the diffusion of electrostatic charge along the surface of a very thin resistor. The collective modes of motion[4] are those of Coulomb particles of zero inertial mass responding to Magnus forces and restricted to move in a single plane. A detailed discussion of such systems will be given elsewhere.

The long-range forces between vortices in thin films, coming mainly from the magnetic interaction in free space, also exist at the surface of bulk samples. Calculations on a semi-infinite slab carrying a vortex line indicate that as the line emerges from the bulk toward free space, the electromagnetic region spreads like a mushroom; the current density at the metal–air interface follows the $1/r^2$ law, while deep inside the metal it falls off exponentially. The range of this transition is of the order of λ. Thus, Abrikosov flux lines are loosely coupled inside the metal but strongly repel each other at their ends, forming a sort of "surface compression layer." Microwave measurements on surfaces perpendicular to the magnetic field are mainly affected by this "stiff" layer, while the internal line lattice remains decoupled.

References

1. A. A. Abrikosov, *Zh. Eksperim. i Teor. Fiz.* **32**, 1442, 1957. (*Soviet Phys. JETP (English Transl.)* **5**, 1174, 1957.)
2. J. Pearl, *Appl. Phys. Letters* **5**, 65, 1964.
3. P. B. Miller, B. W. Kington, and D. J. Quinn, *Rev. Mod. Phys.* **36**, 70, 1964.
4. P. G. de Gennes and J. Matricon, *Rev. Mod. Phys.* **36**, 45, 1964.